ZEROS OF POLYNOMIALS WITH EXTREMECOEFFICIENTS

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Abstract: In this paper, we obtain some interesting extensions and generalizations of well knownEnestrom-Kakeya Theorem.We obtain bounds for the zeros of polynomials by considering restrictions onextreme coefficients.

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Introduction: Many results on the location of zeros of polynomials are available in the literature. Among them the Enestrom and Kakeya theorem [7] given below is well known in the theory of the location of the zeros of polynomials.

Theorem A: Let P(z) $=\sum_0^n a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \text{------} \geq a_1 \geq a_0 > 0 \quad , \quad a_j \in R$$

Then P(z) has all its zeros in the disk $|z|\leq 1$

In the literature [1-8], diverse attempts have been made for generalizing the Enestrom-Kakeya theorem to polynomials and analytic functions. Among others, Gardnerand Govil [4] generalized the Enestrom-Kakeya theorem and proved the following theorem.

Theorem B: Consider an nth-order complex polynomial P(z) $=\sum_0^n a_j z^j$ with Re(a_j) $=\alpha_j$ and Im(a_j) $= \beta_j$, j= 0, 1, 2,...n, and assume that for some k, r and for some t > 0,

$t^n\alpha_n \leq t^{n-1}\alpha_{n-1} \leq \text{----} \leq t^{k+1}\alpha_{k+1} \leq t^k\alpha_k \geq t^{k-1}\alpha_{k-1} \geq \text{----} \geq \alpha_1 \geq \alpha_0$

$t^n\beta_n \leq t^{n-1}\beta_{n-1} \leq \text{----} \leq t^{r+1}\beta_{r+1} \leq t^r\beta_r \geq t^{r-1}\beta_{r-1} \geq \text{----} \geq \beta_1 \geq \beta_0$

Then P(z) has all its zeros in $R_1 \leq |z| \leq R_2$ where

$$R_1 = \min\left\{\frac{t\,|a_0|}{t^n|a_n|^n + 2(t^k\alpha_k + t^r\beta_r) - t^n(\alpha_n + \beta_n) - (\alpha_0 + \beta_0)}, t\right\}$$

and

$$R_2 = \max\Big\{\frac{1}{|a_n|}\Big[t^{n+1}|a_0| - t^{n-1}(\alpha_0 + \beta_0) - t(\alpha_n + \beta_n) + (t^2+1)(t^{n-k-1}\alpha_k + t^{n-r-1}\beta_r) +$$
$$(t^2-1)\Big(\sum_{j=1}^{j=k-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{j=r-1} t^{n-j-1}\beta_j\Big) + (1-t^2)\Big(\sum_{j=k+1}^{j=n-1} t^{n-j-1}\alpha_j + \sum_{j=r+1}^{j=n-1} t^{n-j-1}\beta_j\Big)\Big], \frac{1}{t}\Big\}$$

Choo[3] also generalized EnestromKakeya theorem and proved the following theorem

Theorem C: Let P(z) $=\sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients. If Re(a_j) $=\alpha_j$ and Im(a_j) $= \beta_j$, for j = 0,1,2-----n. such that for some k and r and for some λ , $\mu > 0$

$\lambda t^n\alpha_n \leq t^{n-1}\alpha_{n-1} \leq \text{----} \leq t^{k+1}\alpha_{k+1} \leq t^k\alpha_k \geq t^{k-1}\alpha_{k-1} \geq \text{----} \geq \alpha_1 \geq \alpha_0$

$\mu t^n\beta_n \leq t^{n-1}\beta_{n-1} \leq \text{----} \leq t^{r+1}\beta_{r+1} \leq t^r\beta_r \geq t^{r-1}\beta_{r-1} \geq \text{----} \geq \beta_1 \geq \beta_0$

then P(z) has all its zeros in $R_1 \leq |z| \leq R_2$ where

$R_1 = \frac{|a_0|}{M_1}$ and $R_2 = \frac{M_2}{|a_n|}$ with

$$M_1 = t^n|a_n| + t^n|(\lambda-1)\alpha_n| + t^n|(\mu-1)\beta_n| + 2(t^k\alpha_k + t^r\beta_r) - t^n(\lambda\alpha_n + \mu\beta_n) - (\alpha_0 + \beta_0)$$

and

$$M_2 = t^n|(\lambda-1)\alpha_n| + t^n|(\mu-1)\beta_n| + 2(t^k\alpha_k + t^r\beta_r) - t^n(\lambda\alpha_n + \mu\beta_n) - (\alpha_0 + \beta_0) + |a_0|$$

In this paper we generalize the above theorem and we prove the following theorems.

Theorem 1: Let P(z) $=\sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients. If Re(a_j) $=\alpha_j$ and Im(a_j) $= \beta_j$, for j = 0,1,2-----n. such that for some $0 < \lambda, \mu \leq 1$ andand $0 < \tau, \rho \leq 1$

$\lambda\alpha_n \leq \alpha_{n-1} \leq \text{----} \leq \alpha_{k+1} \leq \alpha_k \geq \alpha_{k-1} \geq \text{----} \geq \alpha_1 \geq \tau\alpha_0$

$\mu\beta_n \leq \beta_{n-1} \leq \text{----} \leq \beta_{r+1} \leq \beta_r \geq \beta_{r-1} \geq \text{----} \geq \beta_1 \geq \rho\beta_0$

where $0 \leq p,q \leq n-1$, then all the zeros of P(z) lie in the disk

$R^{\lambda\mu} \leq |z - z_{\lambda\mu}| \leq R_{\lambda\mu}$,

where $z_{\lambda\mu} = \left[\frac{(1-\lambda)\alpha_n}{a_n} + i\frac{(1-\mu)\beta_n}{a_n}\right]$,

$$R_{\lambda\mu} = \frac{1}{|a_n|}\{2(\alpha_p + \beta_q) - (\lambda\alpha_n + \mu\beta_n) - (\tau\alpha_0 + \rho\beta_0) + (1-\tau)|\alpha_0| + (1-\rho)|\beta_0| + |a_0|\}$$

$$R^{\lambda\mu} = \frac{|a_0|}{|a_n| + |(1-\lambda)\alpha_n| + |(1-\mu)\beta_n| + 2(\alpha_k + \beta r) - (\lambda\alpha_n + \mu\beta_n) - (\tau\alpha_0 + \rho\beta_0) + (1-\tau)\alpha_0 + (1-\rho)\beta_0}$$
$$- \frac{1}{|a_n|}[(1-\lambda)^2\alpha_n^2 + (1-\mu)^2\beta_n^2]^{1/2}$$

Proof: Consider the polynomial

$F(z)= (1-z)P(z) = -a_n z^{n+1} +(a_n - a_{n-1})z^n+\text{--------}+(a_1-a_0)+a_0$

$= [-\alpha_n z^{n+1} +(\alpha_n - \alpha_{n-1})z^n+\text{--------}+(\alpha_1- \alpha_0)z+ \alpha_0] + i[-\beta_n z^{n+1} +(\beta_n - \beta_{n-1})z^n+\text{------} +(\beta_1- \beta_0)z+\beta_0]$

$=\{-\alpha_n z^{n+1} +[(\alpha_n -\lambda\alpha_n)+(\lambda\alpha_n - \alpha_{n-1})]z^n+(\alpha_{n-1}- \alpha_{n-2})z^{n-1}+\text{--------}+ (\alpha_{k+1} - \alpha_k)z^{k+1}+ (\alpha_k - \alpha_{k-1}) z^k+ \text{------}+(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)]z + \alpha_0] +i[-\beta_n z^{n+1}+[(\beta_n -\mu\beta_n) +(\mu\beta_n - \beta_{n-1})]z^n+(\beta_{n-1}- \beta_{n-2})z^{n-1}+\text{-----}+(\beta_{r+1} - \beta_r)z^{r+1}+ (\beta_r - \beta_{r-1})z^r+ \text{------}+[(\beta_1 - \rho\beta_0) + (\rho\beta_0 - \beta_0)]z+\beta_0] \}$

$= -z^n\{(\alpha_n +i\beta_n)z-(1-\lambda)\alpha_n -i(1-\mu)\beta_n\}+[(\lambda\alpha_n - \alpha_{n-1})z^n+(\alpha_{n-1}- \alpha_{n-2})z^{n-1}+\text{----} + (\alpha_{k+1} - \alpha_k)z^{k+1}+ (\alpha_k - \alpha_{k-1}) z^k+ \text{---}+[(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)]z +i[(\mu\beta_n -\beta_{n-1})z^n + (\beta_{n-1}- \beta_{n-2})z^{n-1}+\text{---} +(\beta_{r+1} - \beta_r)z^{r+1}+ (\beta_r - \beta_{r-1})z^r+\text{----}+[(\beta_1 - \rho\beta_0) + (\rho\beta_0 - \beta_0)]z +(\alpha_0 +i\beta_0)]$

Now if $|z|>1$, $\frac{1}{|z|^{n-j}}< 1$, $j= 0,1,2\text{---}n-1$

Therefore,

$|F(z)| \geq |-z|^n\{|a_n z-(1-\lambda)\alpha_n -i(1-\mu)\beta_n|\} - \{|\lambda\alpha_n - \alpha_{n-1}| + \frac{|\alpha_{n-1}-\alpha_{n-2}|}{|z|} +\text{------}+ \frac{|\alpha_{k+1}-\alpha_k|}{|z|^{n-p-1}} + \frac{|\alpha_k-\alpha_{k-1}|}{|z|^{n-k}}+ \frac{|\alpha_{k-1}-\alpha_{k-2}|}{|z|^{n-k+1}}+\text{-----}\frac{|\alpha_1-\tau\alpha_0|}{|z|^{n-1}} + \frac{|1-\tau||\alpha_0|}{|z|^{n-1}}+| \mu\beta_n - \beta_{n-1}| + \frac{|\beta_{n-1}-\beta_{n-2}|}{|z|} +\text{--------}+ \frac{|\beta_{r+1}-\beta_r|}{|z|^{n-r-1}}+ \frac{|\beta_r-\beta_{r-1}|}{|z|^{n-r}}+ \frac{|\beta_{r-1}-\beta_{r-2}|}{|z|^{n-r+1}}+\text{------}+ \frac{|\beta_1-\rho\beta_0|}{|z|^{n-1}} + \frac{|1-\rho||\beta_0|}{|z|^{n-1}}+\frac{|a_0|}{|z|^n} \}$

$= |z|^n\{|a_n z-(1-\lambda)\alpha_n -i(1-\mu)\beta_n|\} - \{2\alpha_p - \lambda\alpha_n - \tau\alpha_0+|a_0|+2\beta_q-\mu\beta_n - \rho\beta_0 + (1-\tau)|\alpha_0| + (1-\rho)|\beta_0|\}>0$,if

$| z -\frac{(1-\lambda)\alpha_n}{a_n}- i\frac{(1-\mu)\beta_n}{a_n}| >\frac{1}{|a_n|} \{2(\alpha_p + \beta_q)-(\lambda\alpha_n+ \mu\beta_n)-(\tau\alpha_0+\rho\beta_0) +(1-\tau)|\alpha_0| + (1-\rho)|\beta_0|+|a_0| \}$

This shows that the zeros of F(z) having modulus greater than 1 lie in

$|z -\frac{(1-\lambda)\alpha_n+i(1-\mu)\beta_n}{a_n})|\leq\frac{1}{|a_n|}\{2(\alpha_p + \beta_q)-(\lambda\alpha_n+ \mu\beta_n)-(\tau\alpha_0+\rho\beta_0) +(1-\tau)|\alpha_0| + (1-\rho)|\beta_0|+|a_0| \}$

Since all the zeros of P(z) with modulus greater than 1 lie in this disc , it can be shown that $R_{\lambda\mu} \geq 1$. Consequently the zeros of P(z) with modulus less than or equal to one are already contained in the disk $|z-z_{\lambda\mu}| \leq R_{\lambda\mu}$

In order to provethe lower bound $R^{\lambda\mu} \leq |z-z_{\lambda\mu}|$ we first prove the following lemma.

Lemma: Let $P(z) =\sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients.Then for $|z|<1$, we show that

$|z| \leq\frac{|a_0|}{M_2} = \frac{|a_0|}{|a_n|+|(\lambda-1)\alpha_n|+|(\mu-1)\beta_n|+2(\alpha_p+ \beta q)-(\lambda\alpha_n+\mu\beta_n)-(\tau\alpha_0+\rho\beta_0)+(1-\tau)\alpha_0+(1-\rho)\beta_0}$

Proof:Let $|z|<1$.

Consider $F(z) = (1-z)P(z)= -a_n z^{n+1} +(a_n - a_{n-1})z^n+\text{--------}+(a_1-a_0)+a_0$

$= [-\alpha_n z^{n+1} +(\alpha_n - \alpha_{n-1})z^n+\text{--------}+(\alpha_1- \alpha_0)z+ \alpha_0] + i[-\beta_n z^{n+1} +(\beta_n - \beta_{n-1})z^n+\text{----} +(\beta_1- \beta_0)z+\beta_0]$

$= -z^n\{(\alpha_n +i\beta_n)z-(1-\lambda)\alpha_n -i(1-\mu)\beta_n\}+\{(\lambda\alpha_n - \alpha_{n-1})z^n+(\alpha_{n-1}- \alpha_{n-2})z^{n-1}+\text{--------}+ (\alpha_{k+1} - \alpha_k)z^{k+1}+(\alpha_k - \alpha_{k-1}) z^k+ \text{------}+ [(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)]z +i[(\mu\beta_n -\beta_{n-1})z^n + (\beta_{n-1}- \beta_{n-2})z^{n-1}+\text{-----}+(\beta_{r+1} - \beta_r)z^{r+1}+ (\beta_r - \beta_{r-1})z^r+\text{-----}+[(\beta_1 - \rho\beta_0) + (\rho\beta_0 - \beta_0)]z] + (\alpha_0 +i\beta_0)$

$= \Psi(z) + a_0,$

Where

$\Psi(z)=-z^n\{(\alpha_n +i\beta_n)z-(1-\lambda)\alpha_n -i(1-\mu)\beta_n\}+\{(\lambda\alpha_n - \alpha_{n-1})z^n+(\alpha_{n-1}- \alpha_{n-2})z^{n-1}+\text{--------}+ (\alpha_{k+1} - \alpha_k)z^{k+1}+(\alpha_k - \alpha_{k-1}) z^k+ \text{------}+ [(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)]z\}+i\{(\mu\beta_n -\beta_{n-1})z^n + (\beta_{n-1}- \beta_{n-2})z^{n-1}+\text{-----}+(\beta_{r+1} - \beta_r)z^{r+1}+ (\beta_r - \beta_{r-1})z^r+\text{-----}+[(\beta_1 - \rho\beta_0) + (\rho\beta_0 - \beta_0)]z\}$

$\therefore| \Psi(z)| = |a_n z-(1-\lambda)\alpha_n-i(1-\mu)\beta_n + \{(\lambda\alpha_n - \alpha_{n-1})z^n+(\alpha_{n-1}- \alpha_{n-2})z^{n-1}+\text{--------}+(\alpha_{k+1} - \alpha_k)z^{k+1} + (\alpha_k - \alpha_{k-1}) z^k+ \text{------} + [(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)]z\}+i\{(\mu\beta_n -\beta_{n-1})z^n + (\beta_{n-1}- \beta_{n-2})z^{n-1}+\text{-----}+(\beta_{r+1} - \beta_r)z^{r+1}+ (\beta_r - \beta_{r-1})z^r+\text{-----}+[(\beta_1 - \rho\beta_0) + (\rho\beta_0 - \beta_0)]z\}|$

$\leq |a_n z-(1-\lambda)\alpha_n-i(1-\mu)\beta_n|+\{2\alpha_k - \lambda\alpha_n - \tau\alpha_0+2\beta_r -\mu\beta_n -\rho\beta_0 + (1-\tau)|\alpha_0| +(1-\rho)|\beta_0|\} \leq | a_n z-(1-\lambda)\alpha_n -i(1-\mu)\beta_n| +M_1$,

$\leq |a_n z|+|(1-\lambda)\alpha_n|+|(1-\mu)\beta_n| +|M_1|$

where $M_1 =2\alpha_k - \lambda\alpha_n - \tau\alpha_0+2\beta_r -\mu\beta_n -\rho\beta_0 + (1-\tau)\alpha_0 + (1-\rho)\beta_0$

Since $\Psi(0)=0$,it follows by Schwarz lemma that

$|\Psi(z)| \le M_1|z|$ for $|z|<1$

Therefore for $|z|<1$,

$|F(z)| = |\Psi(z) + a_o| \ge |a_o|-|\Psi(z)|=|a_o|-|a_n z|-|(1-\lambda)\alpha_n| - |(1-\mu)\beta_n| - M_1|z|$

>0 , if

$|a_o| \ge |a_n z|+|(1-\lambda)\alpha_n|+|(1-\mu)\beta_n| + M_1|z|$

$\ge |z|[\,|a_n|+M_1+\frac{|(1-\lambda)\alpha_n| + |(1-\mu)\beta_n|}{|z|}\,]$

$\ge |z|(|a_n|+|(1-\lambda)\alpha_n|+|(1-\mu)\beta_n|+ M_1)$

$> |z|M_2$,

where

$M_2 = \{|a_n|+|(1-\lambda)\alpha_n|+|(1-\mu)\beta_n|+2(\alpha_k+ \beta_r)-(\lambda\alpha_n + \mu\beta_n)-(\tau\alpha_o+\rho\beta_0)+(1-\tau)\alpha_0 + (1-\rho)\beta_0\}$

Thus, $|z| \le \frac{|a_0|}{M_2}$

Hence P(z) does not vanish in $|z| <\frac{|a_0|}{M_2}$. It can be shown that $M_2 \le |a_o|$ so that $|z|\le 1$. Hence P(z) has all its zeros in $\frac{|a_0|}{M_2} \le |z|$. Now we prove the second part of the main theorem (1)

Since $|z - z_{\lambda\mu}| \ge |z| - |z_{\lambda\mu}|$,

then using eq(15) of above lemma in eq(16), we have

$|z - z_{\lambda\mu}| \ge |z| - |z_{\lambda\mu}| \ge \frac{|a_0|}{M_2} - |z_{\lambda\mu}|$

This implies $\frac{|a_0|}{M_2} - |z_{\lambda\mu}| \le |z - z_{\lambda\mu}|$

$\frac{|a_0|}{M_2} - |\frac{(1-\lambda)\alpha_n}{a_n} + i\frac{(1-\mu)\beta_n}{a_n}| \le |z - z_{\lambda\mu}|$

From above eq we obtain $R^{\lambda\mu} \le |z-z_{\lambda\mu}|$

Hence the above theorem is completely proved.

Corollary 1: Let $P(z) =\sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j) =\alpha_j$ and $Im(a_j) = \beta_j$, for $j = 0,1,2$-----n. such that for some $0 <\lambda \le 1$, and $0<\tau \le 1$

$\lambda\alpha_n \le \alpha_{n-1} \le ----\le\alpha_{p+1} \le \alpha_p\ge\alpha_{p-1}\ge ---- \ge \alpha_1 \ge \tau\alpha_o$

then all the zeros of P(z) lie in the disk

$R^{\lambda\mu} \le |z-z_{\lambda\mu}| \le R_{\lambda\mu}$,

where$z_{\lambda\mu}= \frac{(1-\lambda)\alpha_n}{a_n}$

$R_{\lambda\mu} =\frac{1}{|a_n|}[2\alpha_p- \lambda\alpha_n -\tau\alpha_0 + (1-\tau)|\alpha_0| + |\beta_n| + |\beta_0| +2\sum_{j=0}^{n-1}|\beta_j| + |a_0|\,]$

$R^{\lambda\mu}=\frac{|a_0|}{|a_n|+|(1-\lambda)\alpha_n|+2\alpha_p-\lambda\alpha_n-\tau\alpha_0+(1-\tau)|\alpha_0|+|\beta_n|+|\beta_0|+2\sum_{j=0}^{n-1}|\beta_j|} - \frac{1}{|a_n|}[(1-\lambda)^2{\alpha_n}^2]^{1/2}$

Corollary 2: Let $P(z) =\sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re(a_j) =\alpha_j$ and $Im(a_j) = \beta_j$, for $j = 0,1,2$-----n. such that for some $0 <\mu\le 1$, and $0<\rho \le 1$

$\mu\beta_n \le \beta_{n-1} \le ----\le\beta_{r+1} \le \beta_r\ge\beta_{r-1}\ge ----\ge\beta_1 \ge \rho\beta_o$

then all the zeros of P(z) lie in the disk

$R^{\lambda\mu} \le |z-z_{\lambda\mu}| \le R_{\lambda\mu}$,

where$z_{\lambda\mu}= \frac{(1-\mu)\beta_n}{a_n}$

$R_{\lambda\mu} =\frac{1}{|a_n|}[2\beta_r- \mu\beta_n -\rho\beta_0 + (1-\rho)|\beta_0| + |\alpha_n| + |\alpha_0| +2\sum_{j=0}^{n-1}|\alpha_j| + |a_0|\,]$

$R^{\lambda\mu}=\frac{|a_0|}{|a_n|+|(1-\mu)\beta_n|+2\beta_r-\mu\beta_n-\rho\beta_0+(1-\rho)|\beta\alpha_0|+|\alpha_n|+|\alpha_0|+2\sum_{j=0}^{n-1}|\alpha_j|} - \frac{1}{|a_n|}[(1-\mu)^2{\beta_n}^2]^{1/2}$

Corollary 3 : If in Theorem C, we take$0<\lambda, \mu,\rho,\tau\le 1$ then we obtain the following:

$R_1\le |z| \le R_2$ where$R_1 =\frac{|a_0|}{M_1}$and$R_2 =\frac{M_2}{|a_n|}$with

$M_1 = |\alpha_n| + |(1-\lambda)|\alpha_n| +(1-\mu)|\beta_n| + 2(\alpha_k + \beta_r) -(\lambda\alpha_n+\mu\beta_n) - \tau\alpha_0 + (1-2\tau)|\alpha_0| + (1-2\rho)|\beta_0|$

and

$M_2 = (1-\lambda)|\alpha_n| +(1-\mu)|\beta_n| + 2(\alpha_k + \beta_r) -(\lambda\alpha_n+\mu\beta_n) - (\tau\alpha_0 + \rho\beta_0) +(1-2\tau)|\alpha_0| + (1-2\rho)|\beta_0| + |a_0|$

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