DUALITY FOR A CLASS OF NON SMOOTH GENERALISED B-CONVEX FUNCTIONS

G.V.SARADA DEVI

Abstract: It is well known that convexity play an important role in establishing the sufficient optimality condition and duality theorem for a non linear programming problem. Several class of functions have been defined for the purpose of weakening the limitations of convexity. Recently, the class of convex functions has been extended to the class of B - vex functions by Bector and Singh [3]. In [2], Bector and Suneja define the class of B - vex functions pseudo B - vex and quasi B-vex functions for differentiable numerical functions. The sufficient optimality conditions and duality results were obtained involving these generalized functions. In [62] P. Kariappa and P. Pandian define the B -vexity in non linear programming Duality. In [72], Sudha Gupta define Duality in multi-objective non-linear programming involving semi-locally b - vex and related functions. But no serious attempt made in utilizing the recent developed concepts like optimality and duality for a class of non-smooth generalized B - convex programming.

Definitions and properties of B - convex functions:x be a non - empty subset of Rn, B be an open unit fall in Rn, and let R+ denote the set of non negative real numbers, f : x $\rightarrow$ R+, b: x x x x [0, 1] $\rightarrow$ R+ and b (x, u, λ) is continuous set at λ = 0 for fixed x, u. Suppose that x and y are in Rn. Then

Definition :A real valued function f : x $\rightarrow$ R is said to be Lipschitz at x $\in$ x if there exists two positive constants $\in$ k such that

$| f (z) - f (y) | \leq k || z - y || \ \forall z, y \in x + \in B$

f is said to be locally Lipschitz if f is Lipschitz at every x $\in$ x. Where x + $\in$B = {z || z - x || <$\in$}. For a locally Lipschitz function f, the generalized directional derivative and the generalized sub-differential are define by

fo (x, v) = Sup

δ f(x) = {$\xi\in$ Rn/ fo (x, v) $\geq${ξ , v) $\forall$ v $\in$ Rn}

Definition : The function f is said to B - convex at if is a function such that

f [λ x + (1 - λ) u]$\leq\lambda$ b (x, u, λ) f (x) + (1 - λ b (x, u, λ) f (u)

f is said to be B - convex on X. If it is B - convex at each u $\in$ x.

Using directional derivative, we have

Definition :A local lipschitz function f : x $\rightarrow$ R is said to be (ρ, B, η) - convex at u $\in$ x with respect to some function η, θ : x x x$\rightarrow$ Rn (θ (x, u $\neq$ 0 when x $\neq$ u), if there exists a real numbers ρ such that

b (x, u) [f (x) - f (u)]$\geq$ fo (u, η (x, u) + ρ || θ (x, u) ||2$\forall$ x $\in$ x

'f' is said to be (ρ, B, η) - convex an x, if (5.8) holds each u $\in$ x. If ρ> 0, then f is said to be strongly (B, η) - convex. If ρ = 0, then f is said to be (B, η) - convex. If ρ< 0, then f is said to be weakly (B, η) - convex.

Definition :A locally lipschitz function is said to pseudo (B, η) - convex at u $\in$ x if there exists a function b (x, u) such that

fo(u, η (x, u)) $\geq$ 0 $\Rightarrow$ b (x, u) f (x) $\geq$ b (x, u) f (u) $\forall$ x $\in$ x

f is said to be pseudo (B, η) - convex on x, if it is pseudo (B, η) - convex at each u $\in$ x.

If fo(u, η (x, u)) $\geq$ 0 $\Rightarrow$ b (x, u) f (x) > b (x, u) f (u) $\forall$ x $\in$ x

Then we gain the strictly pseudo definition. (η-convex means convex function with respect to η (x, u).

Definition :A locally Lipschitz function f is said to be quasi (B, η) - convex at u $\in$ x if exists a function b (x, u) such that

f (x) $\leq$ f (u)$\Rightarrow$ b (x, u) f^{o} (u, η (x, u) $\leq$ 0, $\forall$ x $\in$ x

f is said to be quasi (B, η) - convex on x, if it is quasi (B, η) - convex at each u $\in$ x. Lemma : 5.3.1 & Lemma 5.3.2.

Example : Define x = , f : x $\rightarrow$ R as

f (x) = | Sinx | and define b, n : x x x$\rightarrow$ R

as η (x, u) =

b (x, u) =

f is (B, η) - invex. However f is not B - convex because

f [u + $\lambda\eta$ (x, u)]$\leq\lambda$ b (x, u, λ) (f (x) + (1 - λ b (x, u, λ) f (u)

at x = π/3, u = λ/6, λ = 1/2

Example :

Define x = , $f : x \rightarrow R$ as
$f(x) = |x| + |\sin x|$, define b, $\eta : x \times x \rightarrow R$ as
$\eta(x, u) =$, $b(x, u) = 2$ at $u = 0$,
$f^o(0, v) = 2|v|, \forall v \in x$, so f is pseudo
(B, η) - convex at u = 0, but f is not
(B, η) - convex because at $x = \pi/4$
$f^o(0, \eta(x, 0)) > b(x, 0)[f(x) - f(0)]$

Formulation : Let f (x), g, (x), - - - - - gm (x) be locally Lipschitz functions defined on a non-empty open subset $x \subseteq Rn$. Consider the following primal problem and its dual problem (D) : (P) minf (x)

Sto $g(x) \leq 0 \rightarrow$ (5.9)

and (D1) Max f (u)

Sto $0 \in [\partial o f(u) + yt\, \partial o g(u)] \rightarrow$ (5.10)

$yt\, g(u) \geq 0 \rightarrow$ (5.11)

$y \geq 0 \rightarrow$ (5.12)

(D2) : Max [f (u) + yi gi (u)]

Sto $0 \in [\partial^o f(u) + yi\, \partial^o gi(u)] \rightarrow$ (5.13)

$y \geq 0 \rightarrow$ (5.14)

Necessary Condition :

The point (x*, y*) is said to be a critical point of problem (P), if (x*, y*) satisfies the following generalized Kuhn-Tucker.

Conditions :

$0 \in \partial^o f(x^*) + yi\, x^* \partial^o gi(x^*)$

$y^*t\, g(x^*) = 0$

$g(x^*) \leq 0 \rightarrow$ (5.17)

$y^* \geq 0 \rightarrow$ (5.18)

Let xo={ $x \in x / g(x) \leq 0$ } denote the feasible region for problem (P).

Sufficient Conditions :

Let $x^* \in x$ and there exists $y^* \in Rm$ such that (x*, y*) is a critical point for (P)

Let

(i) f be Bo - convex and gi be B_i - convex at x*, for $i \in I = \{ i / g_i(x^*) = 0\}$

(or)

(ii) f be (Bo, η) convex and g_i (B_i, η) - convex with respect to the same η at x* for $i \in I$ wit

$b\_o^*(x, x^*) =$ $bo(x, x^*, \lambda) > 0\ \forall x \in \lambda o$,

$bi^*(x, x^*) =$ $bi(x, x^*, \lambda), \forall x \in \lambda o$.

Then x* is an optimal solution of problem (P).

Proof :

(i) (5.17) yields that $x^* \in xo$, hence

$gi(x) \leq 0 \Rightarrow gi(x^*), \forall x^* \in xo, i \in I$

Using B_i - convexity of gi at x* yields

$g_i o(x, x^*) \leq b\_i^*(x, x^*)[g_i(x) - g_i(x^*)] \leq 0, \forall x, \in xo, i \in I$

This along with (5.18) yields

$yi^*(x, x^*) \leq 0$ (5.19)

(5.17) and lemma (5.3.1) yields

$0 \leq \text{Max} \{<\xi, x> : \xi \in [\partial o f(x^*) + yi^*\, \partial o\, gi(x^*)]\}$

$= \text{Max} \{<\xi, x> : \xi \in \partial o f(x^*)\} + yi^* \max \{<\xi, x> : \xi \in \partial gi(x^*)\}$

$= fo(x, x^*) + yi^*\, g_i o(x, x^*)$ (5.20)

This along with (5.19) yields

$fo(x, x^*) \geq 0$ (5.21)

Using Bo - convexity of f at x* yields

$b\_o^*(x, x^*)[f(x) - f(x^*)] \geq fo(x, x^*) \geq 0$ (5.22)

Thus, from (5.22), it follows that

$f(x) \geq f(x^*), \forall x \in Xo$

Hence, x* is an optimal solution of problem (P).

(ii) It follows on lines of (i)

Duality Theorems:

Weak duality : Assume that x is (P) - feasible and (u, y) is D1 - feasible, Let

(i) f is Bo - convex and gi is Bi convex at u or

(ii) f is (Bo, η) - convex and g_i is (B_i, η) - convex at u with respect to the same η, such that

b_o (x, u) = bo (x, u, λ) > 0

b_i (x, u) = bi (x, u, λ), for all i = 1, 2, ..., m

Then f (x) $\geq$ f (u).

Proof : (i) Since (u, y) is (D1) - feasible, hence

0 $\in \partial$o f (u) + (5.23)

using the lemma 5.3.1, we have

fo (u, x) + (u, x) $\geq 0 \ \forall$ x $\in$ Xo (5.24)

Now, if f (x) < f (u), then the Bo - convexity of f at u yields fo (u, x) < 0 (5.25)

This long with (5.24) yields

(u, x) > 0, $\forall$ x $\in$ Xo (5.26)

This contradicts with g_i is - convex of u for i = 1, 2, ... m. In fact, the convexity of yields.

(u, x) $\leq$ b_i (x, u) [(x) - (u)]

This along with (5.13) and (5.14) yield

(u, x) $\leq$ b_i (x, u) [(x) - (u)] $\leq$

b_i (x, u) (x) $\leq$ 0, $\forall$ x $\in$ Xo

This contrdicts with (5.26), hence

f (x) $\geq$ f (u)

(ii) it follows on the lines of (i)

Strong Duality :

Let x* be (P) - optimal and let g satisty the generalized Kuhn - Tucker constraint at x*. Then, there exists y* $\in$ Rm such that (x*, y*) is (D1) - feasible and the (P) - objective at x* is equal to the (D1) - objective value at (x*, y*) for all D1 - feasible (u, y). Moreover, if f is B° - convex and is - vex at u. (or) (ii) f is (Bo, η) - convex and g_i is

(B, η) - convex at u with respect to the same η such that for all x $\in$ Xo,

b_o (x, u) = bo (x, u, λ) > 0

bi (x, u) = bi (x, u, λ), for all i = 1, 2, ..., m.

Then (x*, y*) is D1 - optimal

Proof : (i) According to the assumption of the theorem, there exists y* $\in$ Rm such that the generalized Kuhn-Tucker condition 5.15 - 5.18 are satisfied (5.15, (5.16), (5.18) yield that (x*, y*) is (D1) - feasible. Also (5.16) yields that the () - objective at x* is equal to the (D1) objective at (x*, y*). Now, if (x*, y*) is not (D1) - optimal, then there exists a (D1) - feasible (x_, y_) such that f (x_) > f (x*).

This contradicts (i) theorem 5.6.1 hence, (x*, y*) is (D1) - optimal.

(ii) it follows on the lines of (i)

For the dual problem (D2). We have the following results.

Theorem : Let the point (x*, y*) is a critical point for (). If f is (o, B, η) - convex and, for each i = 1, 2, ... m, is (i, B, η) - convex with respect to the same functions η, θ and b (x, u) = b (x, u, λ) > 0 and if, (o +) $\geq$ 0, for each feasible point (u, y) of (D2), then x* is global optimal for (P), (x*, y*) is global optimal for (D2), and the optimal values of (P) and (D2) are equal.

Proof : Let x $\in$ xo = {x / g (x) $\leq$ 0) $\leq$ 0} and (u, y) be feasible for (D2). Then, we have (using Lemma 5.1)

fo (u, η (x, u)) + (x, η (x, u))$\geq$ 0, and

g (x) $\geq$ 0

Associate Professor In Mathematics,
A.J Kalasala Machilipatnam,Krishna(Dt),Ap

***