STRONG DOMINATOR COLORING IN GRAPHS

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Abstract: Let G = (V, E) be a graph. A vertex u strongly dominates a vertex v if uv ∈ E and deg(u) ≥ deg(v). A k-strong dominator coloring of a graph G is a proper coloring in which every vertex of G strongly dominates every vertex of at least one color class. The strong dominator chromatic number $\chi_{SD}(G)$ is the minimum positive integer k for which G has a k-strong dominator coloring. In this paper starting with the concept of strong dominator coloring and we present several basic results on this parameter. Also we discuss relations with other graph theoretic parameters and prove that the decision problem strong dominator chromatic number is NP-complete and present some problems for further investigation.

Keywords: Coloring, domination, dominator coloring, strong domination.

Introduction and Motivation: Let G = (V, E) be a simple, finite, undirected and connected graph. For graph theoretic terminology we refer to Chartrand and Lesniak [2]. Graph coloring and domination are two major areas in graph theory that have been well studied. An excellent treatment of domination is given in the book by Haynes et al. [4] and survey papers on several advanced topics on domination are given in the book edited by Haynes et al. [5]. Several variations of coloring have been introduced and studied by many researchers. The book by Jenson and Toft [10] gives an extensive survey of various graph coloring. The applications of graph coloring are from such diverse areas as time-tabling, scheduling, frequency assignment, register allocations, coding theory and resource allocation, etc. There are several variants of graph colorings. List coloring, b-coloring, harmonious coloring, total coloring, sum coloring, rank coloring, complete coloring, rainbow coloring are some of the variants of graph coloring. Dominator coloring and strong dominator coloring are also two such variants of graph coloring and we studied these two colorings.

The open neighborhood of v ∈ V is denoted and defined by N(v) = {u ∈V : uv ∈ E} and the closed neighborhood of v ∈ V is N[v] = N(v) ∪{v}. The degree of a vertex v ∈ V is deg(v) = |N(v)|. A vertex of degree zero in G is called an isolated vertex and a vertex of degree one is a pendant vertex or a leaf of G. The vertex which is adjacent to a pendant vertex is called a support vertex and the edge incident to a pendant vertex is called a pendant edge. For any set S ⊆ V, the induced subgraph < S > is the maximal subgraph of G with vertex set S. Thus two vertices of S are adjacent in < S > if and only if they are adjacent in G. A vertex u ∈ V dominates a vertex v ∈ V if uv ∈ E. A vertex v ∈ V dominates a set S ⊆ V if v dominates every vertex in S. A subset D of V is called a dominating set of G if every vertex in V – D is dominated by at least one vertex in D. A vertex u ∈V strongly dominates a vertex v ∈ V if uv ∈ E and deg(u) ≥ deg(v). Similarly, a vertex u ∈ V weakly dominates a vertex v ∈ V if uv ∈ E and deg(u) ≤ deg(v). A vertex v ∈V strongly (weakly) dominates a set S ⊆ V if v strongly (weakly) dominates every vertex in S. A subset D of V is a strong (weak) dominating set if every vertex v in V – D is strongly (weakly) dominated by at least one vertex in D.

A proper coloring of G is an assignment of colors to the vertices of G in such a way that adjacent vertices receive different colors. A color class is the set of vertices, having the same color. The color class corresponding to a color i is denoted by V_i. Every such color class is an independent set. A dominator coloring of a graph G is a proper coloring in which every vertex of G dominates every vertex of at least one color class. The dominator chromatic number $\chi_d(G)$ is the minimum number of colors required for a dominator coloring of G. The concepts of dominator partition and dominator coloring of a graph were introduced by Hedetniemi et al. [8, 9] and studied further by Gera et al. [6, 7].

The above concepts of independence, domination, strong domination and weak domination give to the following parameter:

i(G) = min{|S| : S is a maximal independent set in G}

γ(G) = min{|S| : S is a dominating set in G}

$\gamma_s(G) = \min\{|S| : S$ is a strong dominating set in $G\}$
$\gamma_w(G) = \min\{|S| : S$ is a weak dominating set in $G\}$.

Our aim in this paper is to introduce strong dominator coloring and study bounds and realization of the strong dominator chromatic number in terms of chromatic number and dominator chromatic number. Also, we present the strong dominator chromatic number for classes of graphs and prove that the decision problem strong dominator chromatic number is NP-complete and finally we pose some problems for further investigation.The following definition which is needed for the subsequent section.

Definition 1.1: The definition of star graph can be generalized to a multi-star $K_n(a_1, a_2, \ldots, a_n)$, which is formed by joining $a_i \geq 1, (1 \leq i \leq n)$ pendant vertices to each vertex v_i of a complete graph K_n with $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. The graph $K_2(a_1, a_2)$, $a_i \geq 1$ is said to be bi-star and it is denoted by B_{a_1,a_2}.

2. Strong Dominator Coloring: In this section, we introduce strong dominator coloring and an overview on known bounds on the strong dominator chromatic number for general graphs is given and we have proved several results involving this parameter.

Definition 2.1: Let G be any graph. A proper k-coloring with partition $V_1 \cup V_2 \cup \ldots \cup V_k$ is said to be strong dominator coloring if every vertex $u \in V$ dominates at least one color class V_i and $\deg(u) \geq \deg(v)$ for all $v \in V_i$. That is every vertex of G strongly dominates every vertex of at least one color class.

The strong dominator chromatic number is the minimum positive integer k for which G is k-strong dominator color partition and is denoted by $\chi_{SD}(G)$. It is convention that if $\{v\}$ is a color class, then v strongly dominates the color class $\{v\}$.

Example 2.2: In Fig. 1, the graph G is depicted with a strong dominator coloring.

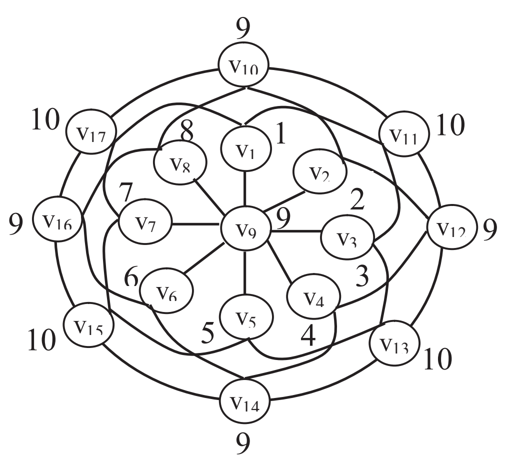

Figure 1: Strong dominator coloring of G

The color classes of G are $V_1 = \{v_1\}$, $V_2 = \{v_2\}$, $V_3 = \{v_3\}$, $V_4 = \{v_4\}$, $V_5 = \{v_5\}$, $V_6 = \{v_6\}$, $V_7 = \{v_7\}$, $V_8 = \{v_8\}$, $V_9 = \{v_9, v_{10}, v_{12}, v_{14}, v_{16}\}$, $V_{10} = \{v_{11}, v_{13}, v_{15}, v_{17}\}$. Therefore $\chi_{SD}(G) = 10$.

Observation 2.3: Let G be a connected graph of order $n \geq 2$. Then at least two different colors are needed in a strong dominator coloring, since there are at least two vertices adjacent to each other. Thus $\chi_{SD}(G) \geq 2$. Every pendant vertex receive unique color for any strong dominator coloring.

Lemma 2.4: For any graph G, $\chi_d(G) \leq \chi_{SD}(G) \leq n$ and the equalities are sharp.

Proof: Let $V = \bigcup_{i=1}^{n} \{v_i\}$ and c be a k-coloring of G. If $c(v_i) = i$, then c is a strong dominator coloring of G, which gives every graph G is strong dominator colorable and implies $\chi_{SD}(G) \leq n$. Every strong dominator color partition of G is a dominator color partition of G implying that $\chi_d(G) \leq \chi_{SD}(G)$. This bounds are sharp as $\chi_{SD}(C_4) = 2$ and $\chi_{SD}(K_n) = n$, $n \geq 2$.

Lemma 2.5 :If G is a regular graph, then $\chi_{SD}(G) = \chi_d(G)$.

Proof: Let G = (V, E) be a k-regular graph. Let C be dominator coloring of G. Clearly $u \in V$ be a vertex dominates a color class V_i and also u strongly dominates V_i. Hence a dominator coloring in a k-regular graph is itself a strong dominator coloring.

Theorem 2.6: Let G be a connected graph of order n. Then $\chi_{SD}(G) = 2$ if and only if $G = K_{a,a}$.

Proof: Let G be a connected graph with $\chi_{SD}(G) = 2$. Then there exist two color classes such that $V(G) = V_1 \cup V_2$. Suppose $|V_1| = 1$ or $|V_2| = 1$, then $G = K_{1,n-1}$, for some $n \geq 2$. This implies, $\chi_{SD}(G) = \chi_{SD}(K_{1,n-1}) = n$, by Observation 2.3. By our assumption $\chi_{SD}(G) = 2$. Hence $G = K_{1,1}$.

Suppose $|V_1| \geq 2$ and $|V_2| \geq 2$. Then every vertex $v \in V_1$ is adjacent to every vertex of V_2 and $\deg(v) \geq \deg(u)$ for every $u \in V_2$. Similarly every vertex in V_2 is adjacent with every vertex in V_1 and $\deg(u) \geq \deg(v)$ for every $u \in V_2$ and $v \in V_1$. Therefore $\deg(u) = \deg(v)$ for every $u \in V_1$ and $v \in V_2$. Hence $G = K_{a,a}$.

Converse is obvious.

Theorem 2.7 : Let G be a graph of order n and every vertex in G be either pendant or support. Then $\chi_{SD}(G) = n$ if and only if every component of an induced subgraph $\langle G - P\rangle$ is a complete graph, where P is the collection of all pendant vertices of G.

Proof: Let P be the collection of pendant vertices of G. Suppose $\chi_{SD}(G) = n$. If $|V(G-P)| \le 2$, then G is a star or bistar. Suppose $|V(G-P)| \ge 3$. If $\langle G-P\rangle$ is not complete, then there exist u, v such that $uv \notin E(G)$ and $\deg(u) \ge \deg(u_i)$, $\deg(v) \ge \deg(v_i)$ where u_i and v_i are pendant neighbours of u and v respectively. Now assign same color for the vertices u and v and also assign unique color for remaining vertices. Clearly, this assignment is strong dominator coloring, hence $\chi_{SD}(G) < n$, which is a contradiction. This implies that $\langle G - P\rangle$ is complete.

Converse is obvious.

Corollary 2.8: For any graph G of order n and every vertex in G is either pendant or support, $\chi_{SD}(G) = n$ if and only if G is isomorphic to multi-star.

Corollary 2.9: Let P be the set of all pendant vertices of graph G. Then $\chi_{SD}(G) \le \chi_{SD}(G - P) + |P|$.

Proof: We can easily observe that, every pendant vertex receive unique color for any strong dominator color partition. Hence $\chi_{SD}(G) \le \chi_{SD}(G - P) + |P|$.

Theorem 2.10: Let G be a bipartite graph. Then $\gamma_w(G) \le \chi_{SD}(G) \le 2 + \gamma_w(G)$ and the inequalities are sharp.

Proof: Let $V(G) = V_1 \cup V_2 \cup \ldots \cup V_k$ be a strong dominator coloring of G. Consider $\chi_{SD}(G) = k$ and $D = \{v_i \in V_i / 1 \le i \le k\}$. Clearly D contains one vertex of each color class. Every vertex in D strongly dominate its own color class and weakly dominates every vertex in V –D, since $\chi_{SD}(G) = k$ and weak domination is reflexive. Thus $\gamma_w(G) \le \chi_{SD}(G)$.

Let D be a minimum weak dominating set of G. Define a coloring $f : V(D) \to \{1, 2, 3, \ldots, \gamma_w(G)\}$ by assigning unique colors to the vertices in D and the remaining vertices are colored by two colors a and b ($a \ne b$) so that no two adjacent vertices receive the same color. Clearly $f(u) \ne f(v)$ for all $uv \in E(G)$ and every vertex in the weak dominating set D provide the color classes that every vertex of G strongly dominates.

The bounds are sharp as $\chi_{SD}(G) = \gamma_w(G)$, for $G = K_{n,n}$, $n \ge 2$ and as $\chi_{SD}(G) = \gamma_w(G) + 2$, for $G = B_{m,n}$, $m, n \ge 1$, where $B_{m,n}$ is the bistar.

Theorem 2.11: Let G be a connected graph and D be a weak dominating set of G. Then $\chi_{SD}(G) \le \gamma_w(G) + \chi(G - D)$.

Proof: Let D be a minimum weak dominating set of G. Now assign unique colors $1, 2, \ldots, \le \gamma_w(G)$ to the vertices in D and assign proper coloring to vertices in G – D, using colors $\gamma_w(G) + 1, \gamma_w(G) + 2, \ldots, \gamma_w(G) + \chi(G - D)$. Clearly, every vertex in G – D strongly dominates a color class i, $1 \le i \le \gamma_w(G)$ and every vertex in D strongly dominates its own color class. The upper bound is sharp for the bistar.

Proposition 2.12: The complete k – partite graph $K_{a_1, a_2, \ldots, a_k}$ ($a_1 \le a_2 \le \ldots \le a_k$) has

$$\chi_{SD}(K_{a_1, a_2, \ldots, a_k}) = \begin{cases} k & \text{if } a_1 \le a_2 \le \ldots \le a_{k-1} = a_k \\ k - 1 + a_k & \text{if } a_1 \le a_2 \le \ldots \le a_{k-1} < a_k. \end{cases}$$

Proof

Let $K_{a_1, a_2, \ldots, a_k}$ be the complete k-partite graph and let $V = V_1 \cup V_2 \cup \ldots \cup V_k$ with $V_i = \{v_{i1}, v_{i2}, \ldots, v_{ia_i}\}$, $1 \le i \le k$ be the k- partite sets, where each V_i is an independent set. Here we observe that for each i, $1 \le i \le k - 1$, $\deg(v_{ij}) \ge \deg(v_{(i+1)j})$, for any j, $1 \le j \le a_i$.

Case (i) $a_1 \le a_2 \le \ldots \le a_{k-1} = a_k$.

We observe that the induced subgraph $\langle\{v_{i1} \in V_i / 1 \le i \le k\}\rangle$ is complete on k vertices and hence we need a minimum of k colors for a proper coloring. Define a coloring f on V such that $f(v_{ij}) = i$, $1 \le i \le k$, $1 \le j \le a_i$.

In this coloring, for $1 \le i \le k - 1$, each vertex $v_{ij} \in V_i$, $1 \le j \le a_i$ strongly dominates the color class i + 1 and each vertex $v_{kj} \in V_k$, $1 \le j \le a_k$ strongly dominates the color class k –1. Hence in this case f is the optimal strong dominator coloring.

Therefore, $\chi_{SD}(K_{a_1,a_2,\ldots,a_k}) = k$ if $a_1 \le a_2 \le \ldots \le a_{k-1} = a_k$.

Case (ii) $a_1 \le a_2 \le \ldots \le a_{k-1} < a_k$.

As in case (i), the coloring f requires k – 1 colors for coloring the first k – 1 partite sets of V. For the vertices in V_k, f is defined as $f(v_{kj}) = k - 1 + j$, $1 \le j \le a_k$. This proper coloring f uses a total of $(k - 1 + a_k)$ colors.

In the following, we justify that the above proper coloring f is the optimal strong dominator coloring of V.

As in case (i), the vertices in V_i, $1 \le i \le k - 1$, strongly dominate the color class i + 1. Each vertex $v_{kj} \in V_k$, $1 \le j \le a_k$ strongly dominates its own color class. If one of the color i, $1 \le i \le k - 1$ is reused at a vertex in V_k, then f ceases to be a proper coloring. On the other hand, if the color at v_{ks}, $1 \le s \le a_k$ is reused at another vertex in V_k, say v_{kt}, $s \ne t$, $1 \le t \le a_k$, then the vertex v_{ks} does not strongly dominate any other color class as $\deg(v_{kj}) < \deg(v_{(k-1)j})$, $1 \le j \le a_k$, implying that f is the optimal strong dominator coloring. Therefore

$$\chi_{SD}(K_{a_1, a_2, \ldots, a_k}) = \begin{cases} k & \text{if } a_1 \le a_2 \le \ldots \le a_{k-1} = a_k \\ k-1+a_k & \text{if } a_1 \le a_2 \le \ldots \le a_{k-1} < a_k. \end{cases}$$

Lemma 2.13:
Let path P_n of order $n \ge 5$, then

$$\chi_{SD}(P_n) \ge \begin{cases} 2+\lceil n/3 \rceil & \text{if } n = 5, 6 \text{ or } n = 3k+1, k \ge 2 \\ 3+\lceil n/3 \rceil & \text{otherwise.} \end{cases}$$

Proof: For n = 5 and 6, verification is straight forward. For $7 \le n-k < n$, by induction hypothesis we assume that

$$\chi_{SD}(P_{n-k}) \ge \begin{cases} 2+\lceil (n-k)/3 \rceil & \text{if } n-k = 3i+1, i \ge 2 \\ 3+\lceil (n-k)/3 \rceil & \text{otherwise.} \end{cases}$$

Let f be a $\chi_{SD}(G)$-coloring of P_n. We assume that the coloring induced on vertices $v_1, v_2, \ldots, v_{n-3}$ is not a strong dominator coloring of P_{n-3}. Then it is obvious that $f(v_{n-3})$ is a reused color. By the Observation 2.3, pendant vertex receives unique color only. So we change the color on v_{n-3} by a new color. Thus a resulting coloring induces a strong dominator coloring of the sub path $v_1, v_2, \ldots, v_{n-3}$.

By the induction hypothesis, at least $2 + \lceil (n-3)/3 \rceil$ colors are used to color vertices $v_1, v_2, \ldots, v_{n-3}$. If f assigns a new color to v_n, then it is a strong dominator coloring and hence the proof is complete. Suppose that f assigns a used color, say i to v_n. Since v_n cannot strongly dominate with color i, $f(v_n)$ is a new color. So f uses at least $2 + \lceil (n-3)/3 \rceil + 1 = 2 + \lceil n/3 \rceil$ colors. Thus $\chi_{SD}(P_n) \ge 2+\lceil n/3 \rceil$ if $n = 3k+1$, $k \ge 2$. Similarly, we can prove that $\chi_{SD}(P_n) \ge 3+\lceil n/3 \rceil$ if $n \ne 3k+1$, $k \ge 2$.

Therefore,

$$\chi_{SD}(P_n) \ge \begin{cases} 2+\lceil n/3 \rceil & \text{if } n = 5, 6 \text{ or } n = 3k+1, k \ge 2 \\ 3+\lceil n/3 \rceil & \text{otherwise.} \end{cases}$$

Proposition 2.14: If path P_n of order $n \ge 2$, then

$$\chi_{SD}(P_n) = \begin{cases} n & \text{when } n = 2, 3 \text{ or } 4 \\ 2+\lceil n/3 \rceil & \text{when } n = 5, 6 \text{ or } n = 3k+1, k \ge 2 \\ 3+\lceil n/3 \rceil & \text{otherwise} \end{cases}$$

Proof: We construct a strong dominator coloring $f : V(P_n) \to \{1, 2, \ldots, \chi_{SD}(P_n)\}$as follows: Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. Let $f(v_1) = 1$. When $n = 3k$, $k \ge 3$, let $f(v_{3i-1}) = 2$, for each i, $1 \le i \le k$. For each i, $1 \le i \le k-1$, let $f(v_{3i}) = 3$ and let $f(v_{3i+1}) = i+3$. Let $f(v_n) = 3+\lceil n/3 \rceil$. When $n = 3k+1$, $k \ge 2$, for each i, $1 \le i \le k$, let $f(v_{3i-1}) = 2$, let $f(v_{3i}) = 3$ and let $f(v_{3i+1}) = i+3$. When $n = 3k+2$, $k \ge 2$, for each i, $1 \le i \le k$, let $f(v_{3i-1}) = 2$, let $f(v_{3i}) = 3$ and let $f(v_{3i+1}) = i+3$. Let $f(v_n) = 3+\lceil n/3 \rceil$. The vertex v_1 colored 1 dominates its own color class, each vertices colored 2 or 3 dominate some uniquely colored neighbor and each vertex colored k for $4 \le k \le 3 + \lceil n/3 \rceil$ dominates its own color class. When n = 2, 3 or 4, it can be easily observed that $\chi_{SD}(P_n) = n$ and when n = 5 or 6, it observed that $\chi_{SD}(P_n) = 2+\lceil n/3 \rceil$.
Hence

$$\chi_{SD}(P_n) \le \begin{cases} n & \text{when } n = 2, 3 \text{ or } 4 \\ 2+\lceil n/3 \rceil & \text{when } n = 5, 6 \text{ or } n = 3k+1, k \ge 2 \\ 3+\lceil n/3 \rceil & \text{otherwise.} \end{cases}$$

By Lemma 2.13, the result is proved.

Observation 2.15 : For any graph G, $\chi(G) \le \chi_d(G)$ $\chi_{SD}(G)$.

Strict inequality as well as equality in observation is possible. By the Corollary 2.8, $\chi(B_{m,n}) < \chi_d(B_{m,n}) < \chi_{SD}(B_{m,n})$, when m, $n \ge 2$. But $\chi(K_{n,n}) = \chi_d(K_{n,n}) = \chi_{SD}(K_{n,n}) = 2$, $n \ge 1$ and so the bound in observation is sharp.

Proposition 2.16: The wheel $W_{1,n}$, $n \ge 3$ has

$$\chi_{SD}(W_{1,n}) = \begin{cases} 1+\lceil n/3 \rceil & \text{when } n = 4 \\ 2+\lceil n/3 \rceil & \text{when } n = 5 \\ 3+\lceil n/3 \rceil & \text{otherwise.} \end{cases}$$

Proof: Let $V(W_{1,n}) = \{v_0, v_1, \ldots, v_n\}$ and $\deg(v_0) = \Delta(W_{1,n})$. We construct a strong dominator coloring $f : V(W_{1,n}) \rightarrow \{1, 2, \ldots, \chi_{SD}(W_{1,n})\}$ as follows: Let $f(v_0) = 1$. When $n = 3k$, $k \ge 1$, for each i, $1 \le i \le k$, let $f(v_{3i-2}) = 2$, let $f(v_{3i-1}) = 3$ and let $f(v_{3i}) = i+3$. When $n = 3k+1$, $k \ge 2$, for each i, $1 \le i \le k$, let $f(v_{3i-2}) = 2$, let $f(v_{3i-1}) = 3$ and let

$f(v_{3i}) = i+3$ and let $f(v_n) = 3+\lceil n / 3 \rceil$. When $n = 3k+2$, $k \geq 2$, for each i, $1 \leq i \leq k+1$, let $f(v_{3i-2}) = 2$. Let $f(v_{3i-1}) = 3$ and let $f(v_{3i}) = i+3$ for each i, $1 \leq i \leq k$. Let $f(v_n) = 3+\lceil n / 3 \rceil$. The vertex v_0 colored 1 strongly dominates its own color class, each vertex colored 2 or 3 strongly dominate some uniquely colored neighbour and each vertex colored j for $4 \leq j \leq 3+\lceil n/3 \rceil$ strongly dominate its own color class. When n = 4 and 5, it can be easily verified that $\chi_{SD}(W_{1,n}) = 1+\lceil n/3 \rceil$ and $\chi_{SD}(W_{1,n}) = 2+\lceil n/3 \rceil$.

Hence $\chi_{SD}(W_{1,n}) \leq \begin{cases} 1+\lceil n/3 \rceil & \text{when } n=4 \\ 2+\lceil n/3 \rceil & \text{when } n=5 \\ 3+\lceil n/3 \rceil & \text{otherwise.} \end{cases}$

On the other hand, in order to reduce the number of colors by 1, if one of the colors used at vertices $\{v_i, 1 \leq i \leq n-1\}$, say color i is reused at v_n, then the vertex v_n do not strongly dominate any color class, implying

$\chi_{SD}(W_{1,n}) \not< \begin{cases} 1+\lceil n/3 \rceil & \text{when } n=4 \\ 2+\lceil n/3 \rceil & \text{when } n=5 \\ 3+\lceil n/3 \rceil & \text{otherwise.} \end{cases}$

3. Complexity Result : Graph k-colorability problem is solvable in polynomial time for many classes, but in general it is NP-complete proved in [3]. In [7], Gera proved DOMINATOR CHROMATIC NUMBER is NP-complete. In this section we prove STRONG DOMINATOR CHROMATIC NUMBER is NP-complete. The reduction is from DOMINATOR CHROMATIC NUMBER.

Decision problem 3.1:

DOMINATOR CHROMATIC NUMBER INSTANCE: Given a graph G and a positive integer k.

QUESTION: Does there exist a function

$f : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that $f(u) \neq f(v)$ whenever $uv \in E(G)$ and for all $v \in V(G)$ there exists a color i such that $\{u \in V(G) : f(u) = i\} \subseteq N[v]$?

Decision problem 3.2:

STRONG DOMINATOR CHROMATIC NUMBER

INSTANCE: Given a graph G and a positive k.

QUESTION: Does there exist a function $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that $f(u) \neq f(v)$ whenever $uv \in E(G)$ and for all $v \in V(G)$ there exists a color i such that $\{u \in V(G) : f(u) = i \text{ and } deg(v) \geq deg(u)\} \subseteq N[v]$?

Theorem 3.3: Strong Dominator Chromatic Number is NP-Complete.

Proof: The decision problem STRONG DOMINATOR CHROMATIC NUMBER is in NP, since we can verify effectively an assignment of colors to the vertices of G is strong dominator coloring.

Consider the instance (G, k), where G is a regular graph with dominator chromatic number k. Let us construct a new instance (G', k') of strong dominator chromatic number as follows: add a vertex v' to G and add an edge from v' to every vertex in G. That is, $V(G') = \{v'\} \cup V(G)$ and $E(G') = \bigcup_{v \in V(G)} \{v', v\} \cup E(G)$.

Set $k' \leftarrow k+1$.

Let us assume that the regular graph G has a dominator coloring with color classes $V_1, V_2, \ldots, V_k$. Since v' is adjacent to every vertex of G, v' is colored by a new color resulting in a dominator coloring with color classes $V_1, V_2, \ldots, V_{k+1}$, where $V_{k+1} = \{v'\}$. By Lemma 2.5, $\chi_{SD}(G) = \chi_d(G) = k$, since G is regular and each vertex $u \in V(G)$ strongly dominates V_i, for some i in G. In G', degree of each vertex $u \in V(G)$ is increased by one and hence $u \in V(G)$ also strongly dominates the color class V_i in G', for some i, using $k' = k + 1$ colors.

Assume that G' possesses a strong dominator coloring using k' colors. It is evident that in our hypothesis v' is the only vertex of its color. Hence by remains v' from G' retains a dominator coloring of G, using $k' - 1 = k$ colors.

Further Research: In this section, we pose some problems for further investigation based on Corollary 2.9, Lemma 2.5 and Theorem 2.10.

1) Characterize graphs for which $\chi_{SD}(G) = \chi_{SD}(G - P) + |P|$, where P is a collection of pendant vertices of a graph G.
2) Characterize graphs G for which $\chi_{SD}(G) = \chi_d(G)$.
3) Characterize bipartite graphs G with $\chi_{SD}(G) = \gamma_w(G)$, $\chi_{SD}(G) = \gamma_w(G)+1$ and $\chi_{SD}(G) = \gamma_w(G) + 2$.

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