ASYMPTOTICALLY QUASI- I- NONEXPANSIVE MAPPING IN BANACH SPACE

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Abstract: In this chapter, I discussed the Asymptotically Quasi- I- Non-expansive mapping in Banach space. Paper explained some definitions for support of the theorems. It is the extension of some results given by others.

Keywords : Asymptotically Quasi- I- Non-expansive mapping, nonself *I*-Asymptotically Quasi Nonexpansive mappings, Nonself Asymptotically Nonsexpansive mapping, retraction, uniformly L-Lipschitzian

Introduction: Let K be a nonempty subset of a real normed linear space X and let $T : K \to K$ be a mapping. Denote by $F(T)$ the set of fixed points of T, that is $F(T) = \{x \in K : Tx = x\}$. Throughout this paper, we always assume that $F(T) \neq \phi$.

Definitions: Let $T : K \to K$, $I : K \to K$ be two mappings of a nonempty subset of K of a real normed linear space X. Then T is said to be

(i) I-nonexpansive, if $\| Tx - Ty \| \le \| Ix - Iy \|$ for all $x, y \in K$;

(ii) asymptotically I-nonexpansive, if there exists a sequence $\{\lambda_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} \lambda_n = 1$ such that $\| T^n x - T^n y \| \le \lambda_n \| I^n x - I^n y \|$ for all $x, y \in K$ and $n \ge 1$;

(iii) asymptotically quasi I-nonexpansive mapping, if there exists a sequence $\{\mu_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} \mu_n = 1$ such that $\| T^n x - p \| \le \mu_n \| I^n x - p \|$ for all $x \in K$, $p \in F(T) \cap F(I)$ and $n \ge 1$.

(iv) Let K be a closed subset of a real Banach space X and let $T : K \to K$ be a mapping ,then

- mapping T is said to be semiclosed (demiclosed) at zero, it for each bounded sequence $\{x_n\}$ in K, the conditions x_n converges weakly to $x \in K$ and Tx_n converges strongly to 0 imply $Tx = 0$.
- mapping T is said to be semicompact, if for any bounded sequence $\{x_n\}$ in K suchthat $\| x_n - Tx_n \| \to 0$, $n \to \infty$, then there exists a subsequence $\{x_{nk}\} \subset \{x_n\}$ such that $x_{nk} \to x^* \in K$ strongly.
- mapping T is called a uniformly L-Lipschitzian mapping, if there exists a constant $L > 0$ such that $\| T^n x - T^n y \| \le L \| x - y \|$ for all $x, y \in K$ and $n \ge 1$.

Theorem 1 : Let C be a nonempty closed convex subset of a Normed linear space X let $T_1, T_2 : C \to X$ be non-self I-Asymptotically Quasi Non-expansive mappings with sequences $\{k_n\}, \{m_n\} \subset [1, \infty)$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$; $\sum_{n=1}^{\infty} (m_n - 1) < \infty$ $I : C \to X$ be a Non-self Asymptotically Non-expansive mapping with sequence $\{l_n\} \subset [1, \infty)$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$. P be a retraction from X onto C. Sequence $\{x_n\}$ is defined as

$$\begin{aligned} x_{n+1} &= P((1-\alpha_n)x_n + \alpha_n I(PI)^{n-1} y_n) \\ y_n &= P((1-\beta_n)x_n + \beta_n T_1 (PT_1)^{n-1} z_n) \\ z_n &= P((1-\gamma_n)x_n + \gamma_n T_2 (PT_2)^{n-1} x_n) \end{aligned}$$

If $F_1 = F(T_1) \cap F(T_2) \cap F(I) \neq \phi$ then

$$\text{Lim}_{n\to\infty} \| x_n - x^* \| \text{ exists for any } x^* \in F.$$

Proof : Let us assume $k_n = 1 + s_n$; $m_n = 1 + t_n$; $l_n = 1 + r_n$.

$$\text{Since } \sum_{n=1}^{\infty} (k_n - 1) < \infty; \ \sum_{n=1}^{\infty} (m_n - 1) < \infty; \ \sum_{n=1}^{\infty} (l_n - 1) < \infty$$

$$\text{so } \sum_{n=1}^{\infty} s_n < \infty; \ \sum_{n=1}^{\infty} t_n < \infty; \ \sum_{n=1}^{\infty} r_n < \infty.$$

Taking $x^* \in F_1$;

$$\begin{aligned}
\| x_{n+1} - x^* \| &= \| \alpha_n I (PI)^{n-1} y_n + (1-\alpha_n) x_n - x^* \| \\
&\le \alpha_n \| I (PI)^{n-1} y_n - x^* \| + (1-\alpha_n) \| x_n - x^* \| \\
&\le \alpha_n (1+r_n) \| y_n - x^* \| + (1-\alpha_n) \| x_n - x^* \| \\
&\le \alpha_n (1+r_n) \| \beta_n (T_1 (PT_1)^{n-1} z_n - x^*) + (1-\beta_n)(x_n - x^*) \| \\
&\qquad + (1-\alpha_n) \| x_n - x^* \| \\
&\le \alpha_n (1+r_n) [\beta_n (1+s_n) \{\| I (PI)^{n-1} z_n - x^* \|\} \\
&\qquad + (1-\beta_n)(x_n - x^*)] + (1-\alpha_n) \| x_n - x^* \| \\
&\le \alpha_n\beta_n (1+r_n)^2 (1+s_n) \| z_n - x^* \| + \alpha_n (1-\beta_n)(1+r_n) \| x_n - x^* \| \\
&\qquad + (1-\alpha_n) \| x_n - x^* \| \\
&\le \alpha_n\beta_n (1+r_n)^2 (1+s_n) \| z_n - x^* \| + [\alpha_n (1-\beta_n)(1+r_n) \\
&\qquad + (1-\alpha_n)] \| x_n - x^* \| \quad \text{...(1)}
\end{aligned}$$

Now

$$\begin{aligned}
\| z_n - x^* \| &= \| (\gamma_n T_2 (PT_2)^{n-1} x_n + (1-\gamma_n) x_n) - x^* \| \\
&\le \gamma_n (1+t_n) \| I (PI)^{n-1} x_n - x^* \| + (1-\gamma_n) \| x_n - x^* \| \\
&\le \gamma_n (1+t_n)(1+r_n) \| x_n - x^* \| + (1-\gamma_n) \| x_n - x^* \|
\end{aligned}$$

putting this value of $\| z_n - x^* \|$ in equation (1) we observe

$$\begin{aligned}
\| x_{n+1} - x^* \| &\le \alpha_n\beta_n (1+r_n)^2 (1+s_n) [\gamma_n (1+t_n)(1+r_n) \| x_n - x^* \| \\
&\qquad + (1-\gamma_n) \| x_n - x^* \|] + [\alpha_n (1-\beta_n)(1+r_n) + (1-\alpha_n)] \| x_n - x^* \| \\
&\le \alpha_n\beta_y\gamma_n (1+s_n)(1+t_n)(1+r_n)^3 \| x_n - x^* \| \\
&\qquad + \alpha_n\beta_n (1-\gamma_n)(1+s_n)(1+r_n)^2 \| x_n - x^* \| \\
&\qquad + [(1-\alpha_n) + \alpha_n (1-\beta_n)(1+r_n)] \| x_n - x^* \| \\
&\le \alpha_n\beta_n\gamma_n (1+s_n)(1+t_n)(1+r_n)^3 \| x_n - x^* \| \\
&\qquad + (\alpha_n\beta_n - \alpha_n\beta_n\gamma_n)(1+s_n)(1+t_n)(1+r_n)^3 \| x_n - x^* \| \\
&\qquad + (1-\alpha_n)(1+r_n)^3 (1+s_n)(1+t_n) \| x_n - x^* \| \\
&\qquad + (\alpha_n - \alpha_n\beta_n)(1+r_n)^3 (1+s_n)(1+t_n) \| x_n - x^* \|
\end{aligned}$$

$$\| x_{n+1} - x^* \| \le (1+r_n)^3 (1+s_n)(1+t_n) \| x_n - x^* \|$$

By our assumption $\sum_{n=1}^{\infty} r_n < \infty; \ \sum_{n=1}^{\infty} s_n < \infty; \ \sum_{n=1}^{\infty} t_n < \infty$

we know that if $\{\alpha_n\}$ and $\{t_n\}$ be two nonnegative sequences satisfying

$$\alpha_{n+1} \le \alpha_n + t_n; \ \forall \ n \ge 1$$

if $\sum_{n=1}^{\infty} t_n < \infty$; then α_n exists. Hence $\lim_{n\to\infty} \| x_n - x^* \|$ exists.

Theorem 2: Let X be a uniformly Banach space. Let C, T_i, I, $\{x_n\}$ be same as above and $\{x_n\}$ also be defined as the same. If T_i is uniformly L-Lipschitzian for some $L > 0$ and $F_1 \ne \phi$ then

$$\lim_{n\to\infty} \| T_i x_n - x_n \| = \lim_{n\to\infty} \| I x_n - x_n \| = 0. \qquad i = 1, 2, 3.$$

Proof : If $x^* \in F$, $\lim_{n\to\infty} \| x_n - x^* \|$ exists, then $\{x_n\}$ is bounded, let

$$\lim_{n\to\infty} \| x_n - x^* \| = c$$

$$\begin{aligned}\| z_n - x^* \| &= \| \gamma_n T_2 (PT_2)^{n-1} x_n + (1-\gamma_n) x_n - x^* \| \\ &= \gamma_n \| T_2 (PT_2)^{n-1} x_n - x^* \| + (1-\gamma_n) \| x_n - x^* \| \\ &\le \gamma_n (1+t_n) \| I (PI)^{n-1} x_n - x^* \| + (1-\gamma_n) \| x_n - x^* \| \\ &\le \gamma_n (1+t_n)(1+r_n) \| x_n - x^* \| + (1-\gamma_n) \| x_n - x^* \| \\ &\le (1+t_n)(1+r_n) \| x_n - x^* \|\end{aligned}$$

on taking lim sup on both sides, we get

$$\limsup_{n\to\infty} \| z_n - x^* \| \le c$$

Now $\| T_1 (PT_1)^{n-1} z_n - x^* \| \le (1+s_n)(1+r_n) \| z_n - x^* \|$

$$\limsup_{n\to\infty} \| T_1 (PT_1)^{n-1} z_n - x^* \| \le c$$

Also $\lim_{n\to\infty} \| \beta_n (T_1 (PT_1)^{n-1} z_n - x^*) + (1-\beta_n)(x_n - x^*) \| = c$

$$\lim_{n\to\infty} \| T_1 (PT_1)^{n-1} z_n - x_n \| = 0$$

$$\Rightarrow \lim_{n\to\infty} \| I (PI)^{n-1} z_n - x_n \| = 0 \qquad ...(3)$$

Now
$$\begin{aligned}\| x_n - x^* \| &\le \| x_n - I (PI)^{n-1} y_n \| + \| I (PI)^{n-1} y_n - I (PI)^{n-1} z_n \| \\ &\quad + \| I (PI)^{n-1} z_n - x^* \| \\ &\le \| x_n - I (PI)^{n-1} y_n \| + (1+s_n)(1+r_n) \| z_n - x^* \|\end{aligned}$$

$$\lim_{n\to\infty} \| z_n - x^* \| = c$$

Again $\| T_2 (PT_2)^{n-1} x_n - x^* \| \le (1+t_n)(1+r_n) \| x_n - x^* \|$

$$\limsup_{n\to\infty} \| T_2 (PT_2)^{n-1} x_n - x^* \| \le c$$

Now $\lim_{n\to\infty} \| z_n - x^* \| = c$ means.

$$\lim_{n\to\infty} \| \gamma_n T_2 (PT_2)^{n-1} (x_n - x^*) + (1-\gamma_n)(x_n - x^*) \| = c$$

$$\lim_{n\to\infty} \| T_2 (PT_2)^{n-1} x_n - x_n \| = 0 \qquad ...(4)$$

And
$$\begin{aligned}\| I (PI)^{n-1} x_n - x_n \| &= \| I (PI)^{n-1} x_n - I (PI)^{n-1} y_n + I (PI)^{n-1} y_n - I (PI)^{n-1} z_n \\ &\quad + I (PI)^{n-1} z_n - x_n \| \\ &\le (1+r_n) \| x_n - y_n \| + (1+r_n) \| y_n - z_n \| + \| I (PI)^{n-1} z_n - x_n \| \\ &\le (1+r_n) \| x_n - z_n \| + \| I (PI)^{n-1} z_n - x_n \| \\ &\le (1+r_n) \| x_n - \{(1-\gamma_n) x_n + \gamma_n T_2 (PT_2)^{n-1} x_n \} \| \\ &\quad + \| I (PI)^{n-1} z_n - x_n \| \\ &\le (1+r_n) \gamma_n \| x_n - T_2 (PT_2)^{n-1} x_n \| + \| I (PI)^{n-1} z_n - x_n \|\end{aligned}$$

From (3) & (4)

$$\lim_{n\to\infty} \| I (PI)^{n-1} x_n - x_n \| = 0 \qquad (5)$$

(4) and (5) implies

$$\lim_{n\to\infty} \| I (PI)^{n-1} x_n - T_2 (PT_2)^{n-1} x_n \| = 0 \qquad ...(6)$$

Now
$$\begin{aligned}x_{n+1} - x_n &\le \alpha_n \| I (PI)^{n-1} y_n - x_n \| \\ &= \alpha_n [\| I (PI)^{n-1} y_n - I (PI)^{n-1} z_n \| + \| I (PI)^{n-1} z_n - x_n \|]\end{aligned}$$

$$\leq \alpha_n (1 + r_n) \| y_n - z_n \| + 0$$

$$\leq \alpha_n (1 + r_n) [\| y_n - x_n \| + \| x_n - z_n \|] \quad (7)$$

So $\| x_{n+1} - z_n \| \leq \| x_{n+1} - I(PI)^{n-1} y_n \| + \| y_n - x_n \| + \| x_n - I(PI)^{n-1} z_n \|$

$$\leq \| x_{n+1} - I(PI)^{n-1} y_n \| + \| \beta_n T_1 (PT_1)^{n-1} z_n - x_n \| + \| x_n - I(PI)^{n-1} z_n \|$$

$$\lim_{n\to\infty} \| x_{n+1} - z_n \| = 0$$

Again $\| x_n - Ix_n \| = \| x_n - I(PI)^{n-1} x_n + I(PI)^{n-1} x_n - I(PI)^{n-1} y_{n-1} + I(PI)^{n-1} y_{n-1} - I(PI)^{n-1} z_{n-1} + I(PI)^{n-1} z_{n-1} - Ix_n \|$

$$\Rightarrow \lim_{n\to\infty} \| x_n - Ix_n \| = 0$$

We have assumed that T_1 is uniformly L-Lipschitzian for some $L > 0$

$$\| T_1 x_n - x_n \| = \| T_1 x_n - T_1 (PT_1)^{n-1} x_n + T_1 (PT_1)^{n-1} x_n - x_n \|$$

$$\leq \| T_1 (PT_1)^{n-1} x_n - x_n \| + \| T_1 (PT_1)^{n-1} x_n - T_1 x_n \|$$

$$\leq \| T_1 (PT_1)^{n-1} x_n - x_n \| + L \| T_1 (PT_1)^{n-2} x_n - x_n \|$$

$$\leq \| T_1 (PT_1)^{n-1} x_n - x_n \| + L \| T_1 (PT_1)^{n-2} x_n - T_1 (PT_1)^{n-2} x_{n-1} \| + L \| T_1 (PT_1)^{n-2} x_{n-1} - x_{n-1} \| + L \| x_{n-1} - x_n \|$$

$$\lim_{n\to\infty} \| T_1 x_n - x_n \| = 0.$$

Similarly we can prove that $\lim_{n\to\infty} \| T_2 x_n - x_n \| = 0.$

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