ALTERNATIVE NUCLEUS OF (-1, 1) RING

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Abstract: If R is a prime finitely generated non alternative 2- torsion free (-1, 1) ring then the alternative nucleus N_β coincides the center C and the ring is alternative.

Keywords: (-1, 1) ring, Nucleus, Alternative nucleus, torsion free ring.

Introduction: Thedy [1] has shown that if R is 2- torsion free simple non alternative ring and satisfies $[R, N_\beta] \subseteq N_\beta$, then N_β coincides with the center of R. Not all simple rings are alternative. Makheev [3] constructed an example of left simple right alternative ring which is not alternative. He also constructed an example of a finite dimensional prime right alternative non alternative ring with a nontrivial idempotent in left nucleus. Nam [5] studied the properties of right nucleus in right alternative algebra free of 2-torsion. He showed that if R is semi-prime and purely non associative, then the nucleus N equals the center C. If further R is finitely generated or free of locally nilpotent ideals, then the right nucleus equals the center. Using the results of [1, 3, 5] in this paper it is shown that in a prime finitely generated non alternative 2- torsion free (-1, 1) ring alternative nucleus N_β coincides the center C and the ring is alternative.

As usual in any non associative ring R the commutator [x, y] and the associator (x, y, z) are defined as xy – yx and (xy)z – x(yz) respectively. R is said to be a 2-torsion free if 2x = 0 implies x = 0 for all x ∈ R. The ring R is said to be (-1, 1) if it satisfies the following two identities:

$(x, y, z) + (x, z, y) = 0$ …(1)

and $(x, y, z) + (y, z, x) + (z, x, y) = 0$ …(2)

for all x, y, z ∈ R.

R is said to be alternative if it satisfies (x, y, y) = 0 = (x, x, y) for all x, y, ∈ R.

Throughout this paper R is assumed to be a 2, 3-torsion free (-1, 1) ring. The right nucleus N_r, the nucleus N, the alternative nucleus N_β and the center C of R are defined as follows:

$N_r : \{n \in R \,/\, (R, R, n) = 0\}$,

$N : \{n \in R \,/\, (n, R, R) = (R, R, n) = 0\}$,

$N_\beta : \{ n \in R \,/\, (x, x, n) = 0 \text{ for all } x \in R\}$,

$C : \{n \in N \,/\, [n, R] = 0\}$.

The following identities hold in a 2-torsion free (-1, 1) ring [5]:

$[[x, y], z] + [[y, z], x] + [[z, x], y] = 2(x, y, z) + 2(y, z, x) + 2(z, x, y)$ …(3)

$[xy, z] – x[y, z] – [x, z]y – (x, y, z) – (z, x, y) + (x, z, y) = 0.$ …(4)

Equation (4) is known as semi Jacobi identity which is valid in any arbitrary nonassociative ring.

$(x, yz, y) = (x, z, y)y$ …(5)

$(x, y^2, z) = (x, y, yz + zy)$ …(6)

$(xy, w, z) = x(y, w, z) + (x, w, z)y – (x, y, [w, z])$ …(7)

$((x, y, z), a, b) = (x, y, (z, a, b)) + (x, (y, a, b), z) + ((x, a, b), y, z) – (x, y, z)[a, b] + (x, y, z[a, b]) – (x, y, [a, b])z$ ... (8)

$[x, (y, y, x)] = (xy, y, x) + (x, xy, y) + (y, x, xy).$ …(9)

It is also known that a 3- torsion free (-1, 1) ring satisfies

$(w, [x, y], z) = ([x, y], z, w) = (z, w, [x, y]).$ …(10)

From (2) and (10) we see that

$(w, [x, y], z) = ([x, y], z, w) = (z, w, [x, y]) = 0.$ …(11)

This implies that $[R, R] \subseteq N$.

Now $x \in N_\beta$ and z = w in (11) gives

$(w, [x, y], w) = ([x, y], w, w) = (w, w, [x, y]) = 0.$

That is (w, w, [x, y]) = 0 and hence $[N_\beta, R] \subseteq N_\beta$. …(12)

Now replacing z, by $w \in N_\beta$ in (11) we also see that

$(w, [x, y], w) = ([x, y], w, w) = (w, w, [x, y]) = 0.$

That is (w, w, [x, y]) = 0 and hence we have (w, w, [R, R]) = 0. That is $[R, R] \subseteq N_\beta$. …(13)

For any $n \in N_\beta$ it was shown in [1] that (x, x, y)n = (x, x, ny). …(14)

Let M be the sub- module of R generated by all alternators (x, x, y). Then M + MR is an ideal of R [1]. Moreover if $W_\beta = \{n \in N_\beta \,/\, Rn \subseteq N_\beta\}$, then the following two theorems were proved by Thedy [1].

Theorem 1: If $[N_\beta, R] \subseteq N_\beta$, then

(i) W_β is an ideal of R;

(ii) $(M + MR)W_\beta = (0)$;

(iii) $[N_\beta, N_\beta] \subseteq W_\beta$, $(N_\beta, N_\beta, N_\beta) \subseteq W_\beta$.

Theorem 2: Let R be a prime finitely generated non associative 2-torsion free right alternative ring. Then the right nucleus N_r coincides with the center of R.

Main Result:

Lemma 1: (i) $(N_\beta, R, R) \subseteq N_\beta$, $(R, R, N_\beta) \subseteq N_\beta$;

(ii) $(R, R, N_\beta) + R(R, R, N_\beta)$ is an ideal of R;

(iii) $\{x, p \in R \,/\, (xp, p, N_\beta) = (x, p, N_\beta p)\, p(x, p, N_\beta), (px, p, N_\beta) = (x, p, N_\beta)p\}$;

(iv) $\{x, p \in R \,/\, (x, R, N_\beta) = 0 = x(R, R, N_\beta)\}$ is an ideal of R.

Proof: (i) From equation (8) with y = x and $z \in N_\beta$, x, a, b ∈ R and using equation (14) we se that

$0 = ((x, x, z), a, b) – (x, x, (z, a, b)) – (x, (x, a, b), z) – ((x, a, b)x, z) + (x, x, z)[a, b] – (x, x, z[a, b]) + (x, x, [a, b])z.$

$= – (x, x, (z, a, b)) – (x, (x, a, b), z) – ((x, a, b), x, z) – (x, x, z[a, b]) + (x, x, z[a, b]).$

$= (x, x, (z, a, b)) + (x, (x, a, b), z) + ((x, a, b), x, z).$

$= (x, x, (z, a, b)) – ((x, a, b), x, z) + ((x, a, b), x, z).$

Thus (x, x, (z, a, b)) = 0.

That is $(N_\beta, R, R) \subseteq N_\beta$. Similarly with $b \in N_\beta$ and applying the same procedure we obtain $(R, R, N_\beta) \subseteq N_\beta$.

(ii) From equation (7) we have (xy, w, z) = x(y, w, z) + (x, w, z)y – (x, y, [w, z]) using equation (12) with $z \in N_\beta$

we obtain $(xy, w, z) = x(y, w, z) + (x, w, z)y$.
That is $(R, R, N_\beta) = R(R, R, N_\beta) + (R, R, N_\beta)R$.
Thus $(R, R, N_\beta)R \subseteq (R, R, N_\beta) + R(R, R, N_\beta)$.
Now
$(R(R, R, N_\beta))R \subseteq (R, (R, R, N_\beta), R) + R((R, R, N_\beta)R)$
$\subseteq (R, R, N_\beta) + R(R, R, N_\beta)$.
Thus $(R, R, N_\beta) + R(R, R, N_\beta)$ is a right ideal of R. By the first part it is also a left ideal.

(iii) Again from equation (7) $(p^2, x, z) = p(p, x, z) + (p, x, z)p - (p, p, [x, z])$.
Now with $z \in N_\beta$ and using equation (12) we see that
$(p^2, x, z) = p(p, x, z) + (p, x, z)p$.
$= p(p, x, z) - (x, p, z)p$
$= p(p, x, z) - (x, p, pz)$ from (14)
and since $(p^2, x, z) = -(x, p^2, z)$
$= -(x, p, pz + zp)$ from (6).
It follows that $p(x, p, z) = (x, p, zp)$. …(15)
From (7) and (5) we obtain $(xp, p, z) = x(p, p, z) + (x, p, z)p - (x, p, [p, z])$
$= (x, p, z)p - (x, p, [p, z])$
$= (x, p, z)p - (x, p, pz) + (x, p, zp)$
$= (x, p, pz) - (x, p, pz) + (x, p, zp)$
$= (x, p, zp)$.
Thus $(xp, p, z) = (x, p, zp) = p(x, p, z)$. …(16)
Again from equation (7) and (5) and the above fact we see that
$(px, p, z) = p(x, p, z) + (p, p, z)x - (p, x, [p, z])$
$= p(x, p, z) + (x, p, [p, z])$
$= p(x, p, z) + (x, p, pz) - (x, p, zp)$
$= (x, p, zp) + (x, p, pz) - (x, p, zp)$ from (15)
$= (x, p, pz)$
$= (x, p, z)p$ from (16).
Thus $(px, p, z) = (x, p, z)p$.

To prove (4) let $W = \{x \in R / (x, R, N_\beta) = 0, x(R, R, N_\beta) = 0\}$ and $x \in W$.
Then $(xR)(R, R, N_\beta) = x(R(R, R, N_\beta))$
$\subseteq x((R, R, N_\beta) + (R, R, N_\beta)R)$ from (7)
$\subseteq (x(R, R, N_\beta))R = (0)$.
and $(Rx)(R, R, N_\beta) = R(x(R, R, N_\beta)) = (0)$.
By the linearized version of part (3) we see that
$(ax, y, N_\beta) \subseteq x(a, y, N_\beta) + (x, y, N_\beta)a - (ay, x, N_\beta) = 0$.
And from equation (7) we have
$(xa, y, N_\beta) = x(a, y, N_\beta) + (x, y, N_\beta)a - (x, a, [y, N_\beta]) = 0$.
Thus W is an ideal of R.♦

Lemma 2 : Let $(R, N_\beta, N_\beta) = 0$, $x, y \in R$. Then
(i) $(N_\beta, (R, R, N_\beta)) = (0)$;
(ii) $(x, R, N_\beta)(x, R, N_\beta) = (0)$;
$[x, N_\beta](x, R, N_\beta) = (0)$;
$(x, x, y)(y, R, N_\beta) = (0)$.

Proof: (i) Let $n_1, n_2 \in N_\beta$ and $x, y, \in R$. Then from equation (7) we obtain
$(x n_1, y, n_2) = x(n_1, y, n_2) + (x, y, n_2)n_1 - (x, n_1, [y, n_2])$
$= (x, y, n_2)n_1$. …(17)
And $(n_1x, y, n_2) = n_1(x, y, n_2) + (n_1, y, n_2)x - (n_1x, [y, n_2])$
$= n_1(x, y, n_2)$. …(18)
(ii) From part (i), $(x, R, N_\beta)(x, R, N_\beta) \subseteq ((x, R, N_\beta)x, R, N_\beta)$ and from Lemma 1 (iii),
$((x, R, N_\beta)x, R, N_\beta) \subseteq ((x, xR, N_\beta), R, N_\beta)$
$\subseteq (N_\beta, R, N_\beta) = (0)$.
Thus $(x, R, N_\beta)(x, R, N_\beta) = (0)$.
It is easy to check that for any $n \in N_\beta$, $x \in R$
Applying equation (4) and (2) with $y = x$ and $z = n \in N_\beta$, we see that
$[x^2, n] - x[x, n] - [x, n]y - (x, x, n) - (n, x, x) + (x, n, x) = 0$.
Therefore $[x^2, n] = x[x, n] + [x, n]x$
$= [x, n]x + [x, [x, n]] + [x, n]x$
$= 2[x, n]x + [x, [x, n]]$.
Consequently, $2[x, N_\beta](x, R, N_\beta) \subseteq 2([x, N_\beta]x, R, N_\beta)$
$\subseteq ([x^2, N_\beta], R, N_\beta) = (0)$.
This proves that $[x, N_\beta](x, R, N_\beta) = (0)$.
From Lemma 1 (iii) and equation (14) we obtain $(x, x, y)(y, R, N_\beta) = (x, x, y(y, R, N_\beta))$
$= (x, x, (y, Ry, N_\beta)) = (0)$. ♦

Lemma 3 : Let R be a prime nonalternative 2- torsion free (-1, 1) ring and $N_\beta \neq N_r$. Then $x(R, R, N_\beta) = 0$ implies $x = 0$.

Proof : From Theorem 1 we have $W_\beta = (0)$. And also $(R, N_\beta, N_\beta) \subseteq W_\beta = (0)$. Let $x \subseteq R$ and $x(R, R, N_\beta) = 0$. Then for any $y \in R$ we have
$(y, y, x)(R, R, N_\beta) \subseteq (y, y, x(R, R, N_\beta)) = (0)$ and so by Lemma 2 (ii) we get
$0 = (y, y, x)(R, R, N_\beta)$
$= -(y, y, R)(x, R, N_\beta)$.
Thus from (14) we see that $(y, y, R(x, R, N_\beta)) = (0)$. Hence $(x, R, N_\beta) \subseteq W_\beta = (0)$. This shows that the set $\{x \in R / x(R, R, N_\beta) = (0)\}$ is contained in the ideal $P = \{x \in R / x(R, R, N_\beta) = 0, (x, R, N_\beta) = 0\}$ from Lemma 1 (iv). Let J be the ideal $(R, R, N_\beta) + R(R, R, N_\beta)$. Then $PJ = (0)$. Thus $P = (0)$ or $J = (0)$. Since $N_\beta \neq N_r$, we have $P = (0)$. This completes the proof of the Lemma. ♦

Lemma 4 : Let R be a prime nonalternative 2-torsion free (-1, 1) ring and $N_\beta \neq N_r$. Then the set $X = \{w \in R / (w, R, N_\beta)(x_1, R, N_\beta)\ldots(x_k, R, N_\beta) = (0)\}$ is a subring of R containing $x_1, x_2, \ldots x_k$.

Proof: Since R is prime and nonalternative, we have $W_\beta = 0$ and also $(R, N_\beta, N_\beta) \subseteq W_\beta = (0)$, $[N_\beta, N_\beta] = (0)$. For convenience, we omit the paranthesis in any expression where we can apply $(R, N_\beta, N_\beta) = (0)$. It is clear by Lemma 2 (ii) that X contains the elements $x_1, x_2, \ldots, x_k$.
Let $w_1, w_2 \in X$ and $P = (x_1, R, N_\beta) \ldots (x_k, R, N_\beta)$. But from (6)
$(w_1w_2, R, N_\beta)P \subseteq (w_1(w_2, R, N_\beta))P + ((w_1, R, N_\beta)w_2)P - (w_1, w_2, [R, N_\beta])P$
$\subseteq ((w_1, R, N_\beta)w_2)P$
$\subseteq (w_1, R, N_\beta)(w_2P)$.
We now prove that $(w_1, R, N_\beta)(w_2P) = (0)$ we have by (14) that
$0 = (w_2, R, N_\beta)P = (w_2P, R, N_\beta)$.
From Lemma 2 (ii), it follows that for any $x, y \in R$
$0 = (x, x, y)(w_2P, R, N_\beta)$
$= -(x, x, w_2P)(y, R, N_\beta)$.
Thus from Lemma 3, we have $(x, x, w_2P) = (0)$.
This shows that $w_2P \subseteq N_\beta$.
Now by Lemma 2 (i) we have
$(w_1, R, N_\beta)(w_2P) = (w_2P)(w_1, R, N_\beta)$

$= w_2(P(w_1, R, N_\beta))$
$= w_2((w_1, R, N_\beta)P)$
$= (0)$.

Therefore X is a subring of R. ♦

Theorem 3: Let R be a prime finitely generated non alternative 2-torsion free (-1, 1) ring. Then N_β concides with the center of R.

Proof : Let R be finitely generated by $x_1, x_2, \ldots, x_k$ and x $= \{w \in R / (w, R, N_\beta)(x_1, R, N_\beta) \ldots (x_k, R, N_\beta) = (0)\}$. Assume that $N_r \neq N_\beta$. Then from Lemma 4, X is a sub-ring of R consisting $x_1, x_2, \ldots, x_k$. Since R is finitely generated by $x_1, \ldots, x_k$, we have X = R and so $(R, R, N_\beta)(x_1, R, N_\beta)\ldots(x_k, R, N_\beta) = (0)$. From Lemma 3, we then have $(x_1, R, N_\beta)\ldots(x_k, R, N_\beta) = (0)$. By making use of induction, we deduce that $(R, R, N_\beta) = (0)$. Hence we must have $N_r = N_\beta$. By Theorem 2, N_β coincides with the center C of R. Therefore $N_\beta = N_r = C$. ♦

Theorem 4 : Let R be a prime finitely generated 2-torsion free (-1, 1) ring. Then R is alternative.

Proof : Assume that R is nonalternative. Then by Theorem 3, N_β coincides with the center C of R. From (13) we see that $[R, R] \subseteq N_\beta \subseteq C$.

Consequently (9) gives $[x, y]^2 = [x, y][x, y]$
$= [x, y[x, y]]$
$= -[x, (y, x, y)]$
$= (0)$.

This proves that [R, R] = 0. Being commutative and (-1, 1), R is alternative. Therefore R must be alternative.

Conclusion: There are many results where we can see that the nucleus is equal to the center in any type of alternative rings. But a ring which is finitely generated and a 2-torsion free nonalternative (-1, 1) becomes an alternative ring when the alternative nucleus coincides the center.

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