ON SEARCHING SHORTEST PATH IN A CLASSICAL NETWORK WITH FUZZY EDGE WEIGHTS

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Abstract: A novel approach is made in this paper to investigate the shortest path network problem in a fuzzy environment. Here the edge weights of a network are not known exactly and only the approximate values are given in terms of trapezoidal fuzzy numbers. New defuzzification formulae are introduced for trapezoidal fuzzy numbers and three different algorithms are presented for fuzzy shortest path problem (FSPP). Examples are illustrated along with a simulation result to demonstrate the proposed approach. The presented method is compared with the existing method which reveals that the method proposed in this paper is more effective in determining the fuzzy shortest path in a network.

Keywords: Network, Shortest path problem, Fuzzy numbers, Ranking measures, Decision making.

Introduction: The shortest path problem concentrates on finding the path with minimum distance. It is one of the basic problems in networks and is widely applied in transportation, communication and computer network. The Bellman algorithm is one of the efficient algorithms used to determine the shortest path in a crisp network. But in real world applications uncertainty cannot be avoided and usually the edge weights (or arc lengths) of a network cannot be determined precisely. For instance, on road networks, because of several reasons like traffic, accidents etc., edge weights representing the vehicle travel time (or cost) are subjected to uncertainty. One way to deal with these types of uncertainties is to utilize probability theory. However, sometimes the probability distributions of the lengths of arcs are difficult to acquire, due to the lack of historical data and hence it can be viewed as fuzzy. The concept of fuzzy set theory introduced by Zadeh in 1965 plays a vital role to deal with the issue of these types of uncertainty problems.

Literature Review*:* The fuzzy shortest path problem was first analyzed by Dubois and Prade [4]. They considered the extension of the classic Floyd and Ford-Moore-Bellman (FMB) algorithms. By their approach, the shortest path length could be obtained but the corresponding path in the network might not exist. Klein [5] proposed a dynamic programming recursion-based fuzzy algorithm. Lin and Chen [8] found the fuzzy shortest path length in a network by means of a fuzzy linear programming approach. Another algorithm for this problem was presented by Okada and Gen [11,12] where there is a generalization of Djikstra's algorithm. Okada [14] introduced the concept of the degree of possibility of an arc being on the shortest path. Chuang and Kung [3] proposed a fuzzy shortest path length among all possible paths in a network. Nayeem and Pal [10] proposed an algorithm based on the acceptance index introduced by Sengupta and Pal [16] which gave a single fuzzy shortest path or a guideline for choosing the best fuzzy shortest path according to the decision maker's viewpoint. To overcome the short-coming of the existing algorithm [10], Kumar and Kaur [6] developed a new algorithm for finding the fuzzy shortest path and fuzzy shortest distance of each node from source node. Thus numerous papers have been published in FSPP.

Outline of the work*:* The paper is developed as follows: Some basic concepts of fuzzy numbers are presented and have coined new defuzzification formulae related to trapezoidal fuzzy numbers, which is utilised to identify the fuzzy shortest path in a network. Three different algorithms are proposed for FSPP. Examples are illustrated along with a simulation result to explain the development of the algorithms. Finally the main conclusion is pointed out.

Pre-requisites:

Definition 1: Triangular fuzzy number [7]

A triangular fuzzy number is represented by a triplet $A=(a,b,c), a,b,c \in \Re^+$ with the membership function

$$\mu_A(x)=\begin{cases}\dfrac{x-a}{b-a} & , \quad a<x<b \\ 1 & , \quad x=b \\ \dfrac{c-x}{c-b} & , \quad b<x<c \\ 0 & , \quad otherwise\end{cases}$$

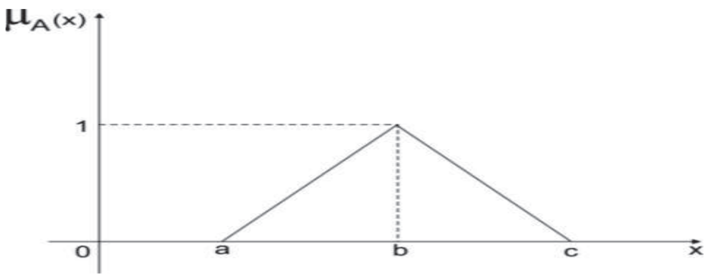


Fig. 1: Triangular fuzzy number A

Definition 2: Centroid Method for triangular fuzzy number [15]

Let $A=(a,b,c)$ be a triangular fuzzy number.

Then $\text{Centroid Method}(A)=\dfrac{a+b+c}{3}$

Definition 3: Acyclic Digraph

A digraph is a graph each of whose edges are directed. An acyclic digraph is a directed graph without cycle.

Definition 4: Crisp graph with fuzzy weights [1]

A fifth type (Type V) of graph fuzziness occurs when the graph has known vertices and edges, but unknown weights on the edges. Thus only the weights are fuzzy.

Application: To plan the quickest automobile route from one city to another.

Definition 5: Trapezoidal fuzzy number and its arithmetic operation [7]

A trapezoidal fuzzy number is represented by a quadruplet $A=(a,b,c,d), a,b,c,d \in \Re^{+}$ with the membership function

$$\mu_A(x)=\begin{cases} \dfrac{x-a}{b-a} & , \; a<x<b \\ 1 & , \; b<x<c \\ \dfrac{d-x}{d-c} & , \; c<x<d \\ 0 & , \; otherwise \end{cases}$$

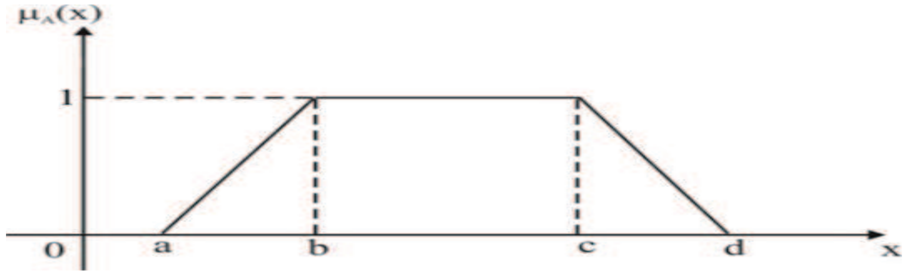


Fig. 2: Trapezoidal fuzzy number

If $A=(a_1,b_1,c_1,d_1)$ and $B=(a_2,b_2,c_2,d_2)$, then $A(+)B=(a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2)$.

Definition 6: Minimum Operation [9]

For two trapezoidal fuzzy numbers, $A=(a_1,b_1,c_1,d_1)$ and $B=(a_2,b_2,c_2,d_2)$,

$$L_{\min}=\left(\min(a_1,a_2),\min(b_1,b_2),\min(c_1,c_2),\min(d_1,d_2)\right)$$

The following definitions are introduced in this paper.

Definition 7: Centroid Measure (CM)

Let $A=(a,b,c,d)$, be a trapezoidal fuzzy number. Then centroid measure of A is defined by

$$CM(A)=\left[\frac{\int_a^b \frac{x-a}{b-a}.xdx+\int_b^{e'} \frac{e'-x}{e'-b}.xdx}{\int_a^b \frac{x-a}{b-a}dx+\int_b^{e'} \frac{e'-x}{e'-b}dx}\right] + \left[\frac{\int_{e'}^c \frac{x-e'}{c-e'}.xdx+\int_c^d \frac{d-x}{d-c}.xdx}{\int_{e'}^c \frac{x-e'}{c-e'}dx+\int_c^d \frac{d-x}{d-c}dx}\right] + \left[\frac{-\int_{e'}^c \frac{x-c}{e'-c}.xdx-\int_b^{e'} \frac{b-x}{b-e'}.xdx}{-\int_{e'}^c \frac{x-c}{e'-c}dx-\int_b^{e'} \frac{b-x}{b-e'}dx}\right]$$

$$CM(A)=\frac{a+2b+2c+d+3e'}{3} \quad \text{where } e'=\frac{b+c}{2}$$

$$=\frac{1}{6}\left(2(a+d)+7(b+c)\right)$$

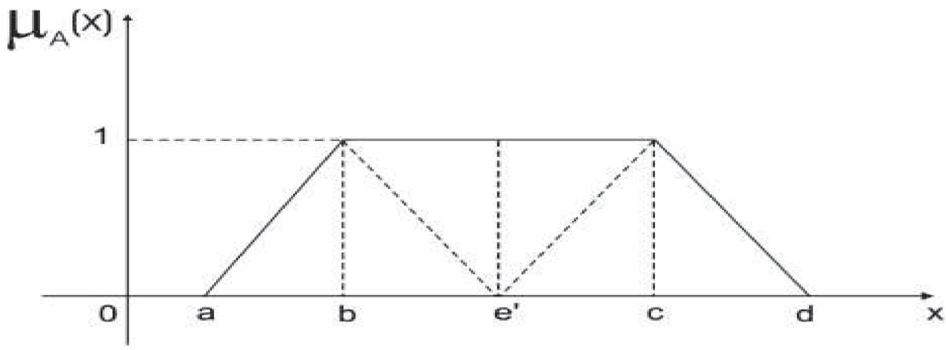


Fig. 3: Trapezium divided into triangles

If A and B are two trapezoidal fuzzy numbers then $A < B$ if and only if $CM(A) < CM(B)$

Definition 8: Alphacut Measure

Let $A = (a,b,c,d)$ be a trapezoidal fuzzy number, $\alpha \in [0,1]$. Then Alphacut measure of A is defined by

$$\alpha - cut\ M(A)\ or\ AlphacutM(A) = \left[\int_0^1 [a + \alpha(b-a)]d\alpha + \int_0^1 [e' + (b-e')\alpha]d\alpha\right] + \left[\int_0^1 [e' + \alpha(c-e')]d\alpha + \int_0^1 [d + (c-d)\alpha]d\alpha\right] + \left[-\int_0^1 [c + \alpha(e'-c)]d\alpha - \int_0^1 [b + (e'-b)\alpha]d\alpha\right] \text{ (Refer Fig.3)}$$

$$= \frac{a+b+c+d}{2}$$

If A and B are two trapezoidal fuzzy numbers then $A < B$ if and only if α-cutM(A)< α-cutM(B)

Definition 9: Mean-Standard Deviation Measure $[(Mean - SD)M\ or\ (\mu - \sigma)M]$

According to [2] a trapezoidal fuzzy number $A = (a,b,c,d)$ can be approximated as a symmetry fuzzy number $S[\mu,\sigma]$, μ denotes the mean of A, σ denotes the standard deviation of A, and the membership function of A is defined as follows:

$$\mu_A(x) = \begin{cases} \frac{x - (\mu - \sigma)}{\sigma}, & if\ \mu - \sigma \le x \le \mu \\ \frac{(\mu + \sigma) - x}{\sigma}, & if\ \mu \le x \le \mu + \sigma \end{cases}$$

where μ and σ are calculated as follows:

$$\sigma = \frac{2(d-a) + c - b}{4}, \mu = \frac{a+b+c+d}{4}$$

Using the above result, the Mean-Standard deviation measure of A is defined by

$$(\mu - \sigma)M(A) = \int_0^1 [(\mu - \sigma) + \sigma\alpha]d\alpha + \int_0^1 [(\mu + \sigma) - \sigma\alpha]d\alpha = 2\mu = \frac{a+b+c+d}{2}$$

which is same as Alphacut Measure. Therefore, if A and B are two trapezoidal fuzzy numbers, then $A < B$ if and only if $(\mu - \sigma)M(A) < (\mu - \sigma)M(B)$

Definition 10: Acceptability Measure (or) Mean-Width notation

Let the i^{th} fuzzy path length be trapezoidal fuzzy number $L_i = (a_i, b_i, c_i, d_i), i = 1\ to\ n$ and the fuzzy shortest length be $L_{\min} = (a,b,c,d), a \le a_i, b \le b_i, c \le c_i, d \le d_i$. Then the Acceptability measure between L_i and $L_{\min}$ is defined as $A(L_{\min}, L_i) = \frac{m_i - m}{(d-a) + (d_i - a_i)}$, $m_i = \frac{a_i + b_i + c_i + d_i}{4}$, $m = \frac{a+b+c+d}{4}$. If A and B are two trapezoidal fuzzy numbers, then $A < B$ if and only if $A(L_{\min}, A) < A(L_{\min}, B)$

Definition 11: Intersection Area Measure

According to [18], if the i^{th} fuzzy path length is $L_i = (a_i, b_i, c_i, d_i), i = 1\ to\ n$ and the fuzzy shortest length is $L_{\min} = (a,b,c,d)$, where $a_i < b_i < c_i < d_i, a < b < c < d$, $a \le a_i, b \le b_i, c \le c_i, d \le d_i$, then for $c < x < d, a_i < x < b_i$, the height of the intersection area is defined as $h = y_d = \frac{d - a_i}{(d-c) + (b_i - a_i)}$.

Here this is taken as Intersection area measure denoted by IAM($L_{\min}$,L_i) which is utilized to identify the shape of the intersection area between $L_{\min}$ and L_i and also to know the various types of dominance between $L_{\min}$ and L_i. Let

$$IAM(L_{\min}, L_i) = \frac{d - a_i}{(d-c) + (b_i - a_i)}$$

- If $d < a_i$, $IAM(L_{\min}, L_i) < 0$ then L_i is said to be very totally dominated by $L_{\min}$. Here $L_{\min} \cap L_i = \Phi$
- If $d = a_i$, $IAM(L_{\min}, L_i) = 0$ then L_i is said to be totally dominated by $L_{\min}$. Here the intersection area between $L_{\min}$ and L_i is a point $d = a_i$.
- If $d > a_i, 0 < IAM(L_{\min}, L_i) \le 1$, then L_i is said to be partially dominated by $L_{\min}$. Here the intersection area between $L_{\min}$ and L_i is in the form of triangle.
- If $d > a_i, IAM(L_{\min}, L_i) > 1$ then L_i is said to be partially dominated by $L_{\min}$. But here the intersection area between $L_{\min}$ and L_i is in the form of trapezium.
- If $a = a_i, b = b_i, c = c_i, d = d_i$, $IAM(L_{\min}, L_i) >$ 1, then L_i is said to be non-dominated by $L_{\min}$. Here the intersection area between $L_{\min}$ and L_i is in the form of trapezium.

Proposed Algorithms:

Algorithm for FSPP based on CM or α - Cut M / (Mean – SD)M

Step 1: (i) Construct an acyclic network (graph) $G(V,E)$ where V is the set of vertices or nodes and E is the set of edges or arcs. (ii) Let $N=\{1,2,3,....,n\}$ be the set of all nodes in a network, $NP(j)$ be the set of all predecessor nodes of node j, K_j be the distance between node j and source node. e_{ij} is the distance between node i and node j (edge weights).

Step 2: Edge weights e_{ij} in terms of trapezoidal fuzzy numbers are converted to any one of the measures namely centroid measure or alphacut measure / mean-standard deviation measure namely e_{ij}^{*}

Step 3: Set $K_1=0$, calculate $K_j=\underset{i<j}{Min}\{K_i \oplus e_{ij}^{*}\}$, $j=2\ to\ n$ where K_j gives the shortest distance between node j and source node.

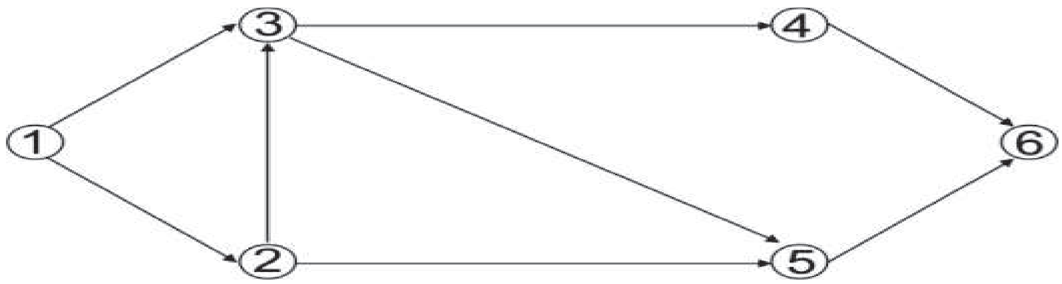


Fig. 4: A classical network

Illustrative Example 1: Step 1: A classical network with trapezoidal fuzzy edge weights is shown in Fig.4

e_{12}=A[1-2]=(10,20,20,30),
e_{13}=B[1-3]=(52,62,65,70),
e_{23}= C[2-3]=(35,38,40,45),
e_{25}=D[2-5]=(52,55,60,65),
e_{34}=E[3-4]=(10,13,17,20),
e_{35}=F[3-5]=(8,9,9,10),
e_{46}=G[4-6]=(70,75,85,97),
e_{56}=H[5-6]=(50,70,80,100).

Step 2: Edge weights in terms of trapezoidal fuzzy numbers are converted into centroid measure using definition 7.

A[1-2]=60 $=e_{12}^{*}$, B[1-3]=188.83=e_{13}^{*},

C[2-3]=117.67= e_{23}^{*}, D[2-5]=173.17=e_{25}^{*},

E[3-4]=45=e_{34}^{*}, F[3-5]=27=e_{35}^{*},

G[4-6]=242.33=e_{46}^{*}, H[5-6]=225=e_{56}^{*}.

Step 3:

$$K_1=0, K_2=K_1+e_{12}^{*}=60,$$

$$K_3=Min\{K_1+e_{13}^{*}, K_2+e_{23}^{*}\}=177.67,$$

$$K_4=K_3+e_{34}^{*}=222.67,$$

$$K_5=Min\ \{K_3+e_{35}^{*}, K_2+e_{25}^{*}\}=204.67,$$

$$K_6=Min\ \{K_4+e_{46}^{*}, K_5+e_{56}^{*}\}=429.67$$

Thus, $K_6=K_5+e_{56}^{*}=K_3+e_{35}^{*}+e_{56}^{*}$

$$=K_2+e_{23}^{*}+e_{35}^{*}+e_{56}^{*}$$

$$=K_1+e_{12}^{*}+e_{23}^{*}+e_{35}^{*}+e_{56}^{*}=e_{12}^{*}+e_{23}^{*}+e_{35}^{*}+e_{56}^{*}$$

Therefore the fuzzy shortest path is $1\rightarrow 2\rightarrow 3\rightarrow 5\rightarrow 6$ and the fuzzy shortest path length is 429.67.

Alternative algorithm for FSPP based on CM , α - Cut M / (Mean – SD)M

Step 1: Construct an acyclic network $G(V,E)$ where V is the set of vertices or nodes and E is the set of edges or arcs.

Step 2: (i) Calculate all possible paths P_i and the corresponding path lengths $L_i, i=1\ to\ n$ using definition 5. (ii) Calculate centroid measure, alphacut measure / mean-standard deviation measure for each path lengths L_i. (iii) The path having the lowest centroid measure, alphacut measure / mean-standard deviation measure is identified as the fuzzy shortest path (FSP). The corresponding path length is the fuzzy shortest path length.

Illustrative Example 2:

Step 1: Construct a classical network with trapezoidal fuzzy edge weights as shown in Fig. 4

Step 2: Refer Table 1

Table 1: Results of the Network				
Paths P_i	Path lengths L_i	$CM(L_i)$	AlphacutM(L_i)/ (Mean-SD)M(L_i)	Ranking
$1\to2\to5\to6$	(112,145,160,195)	458.17	306.00	3
$1\to2\to3\to4\to6$	(125,146,162,192)	465	312.50	4
$1\to2\to3\to5\to6$	(103,137,149,185)	429.67	287.00	1
$1\to3\to4\to6$	(132,150,167,187)	476.17	318.00	5
$1\to3\to5\to6$	(110,141,154,180)	440.83	292.50	2

Path $P_3 = 1\to2\to3\to5\to6$ is the fuzzy shortest path and the corresponding fuzzy shortest path length is L_3=(103,137,149,185)

Algorithm for FSPP based on Acceptability Measure

Step 1: Construct an acyclic network $G(V,E)$ where V is the set of vertices or nodes and E is the set of edges or arcs.

Step 2: (i) Calculate all possible paths P_i and the corresponding path lengths $L_i, i=1\ to\ n$ using definition 5. (ii) Calculate L_{min} using definition 6. (iii) Calculate acceptability measure for each path lengths $L_i, i=1\ to\ n$ and L_{min}. (iv) The path having the lowest acceptability measure is identified as the fuzzy shortest path. (v) Calculate the Intersection area measure between L_{min} and $L_i, i=1\ to\ n$ which gives the shape of the intersection area between them.

Illustrative Example 3:

Step 1: Construct a classical network with trapezoidal fuzzy edge weights as shown in Fig. 4

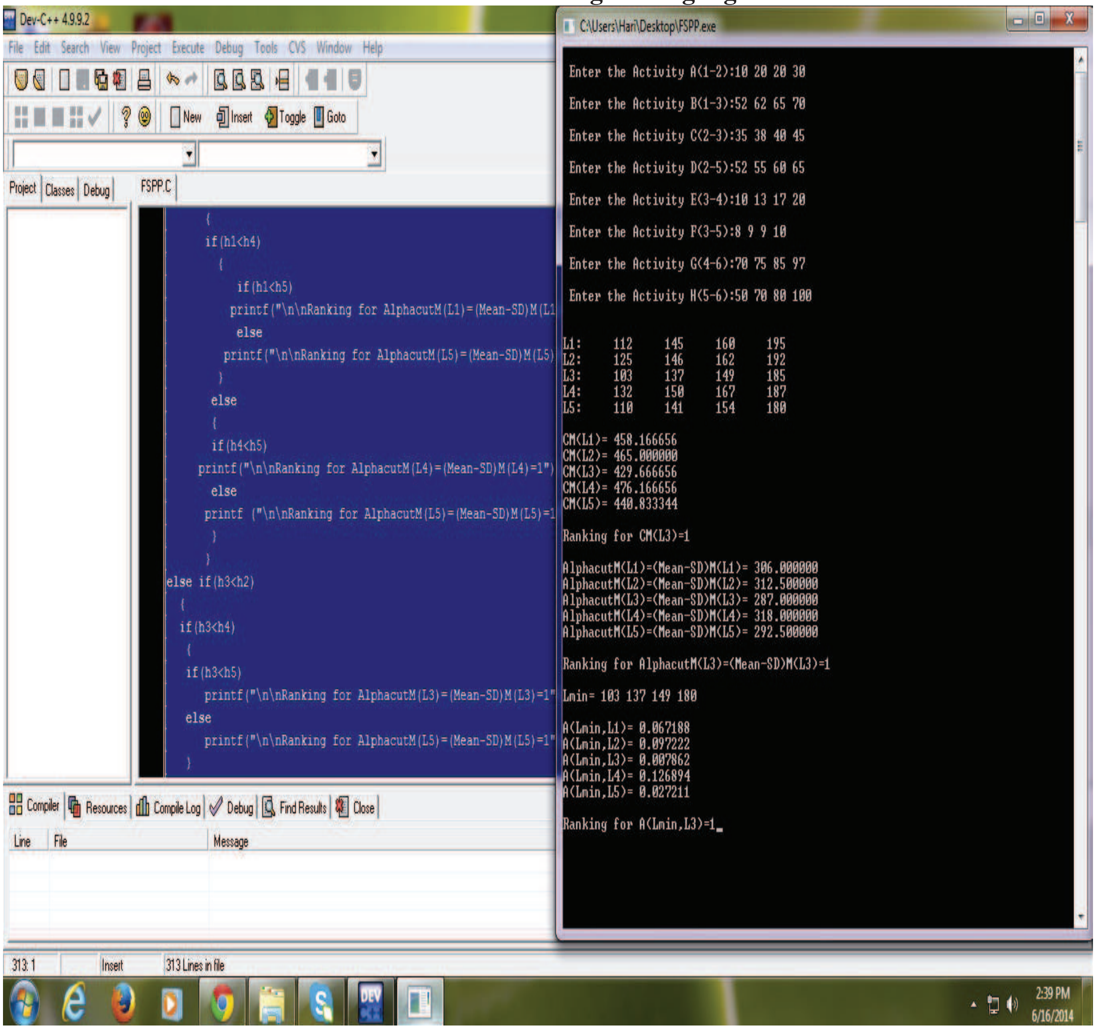
Step 2: Refer Table 2 and Table 3

Table 2: Results of the network			
Paths P_i	Path lengths L_i	$A(L_{min}, L_i)$, L_{min} = (103,137,149,180)	Ranking
$1\to2\to5\to6$	(112,145,160,195)	0.067	3
$1\to2\to3\to4\to6$	(125,146,162,192)	0.097	4
$1\to2\to3\to5\to6$	(103,137,149,185)	0.008	1
$1\to3\to4\to6$	(132,150,167,187)	0.127	5
$1\to3\to5\to6$	(110,141,154,180)	0.027	2

Path $P_3 = 1\to2\to3\to5\to6$ is the fuzzy shortest path.

Table 3: Results of the network based on Intersection area measure			
Paths P_i	Path lengths L_i	$IAM(L_{min}, L_i)$ L_{min} = (103,137,149,180)	Shape of the Intersection Area
$1\to2\to5\to6$	(112,145,160,195)	1.063	Trapezium
$1\to2\to3\to4\to6$	(125,146,162,192)	1.058	Trapezium
$1\to2\to3\to5\to6$	(103,137,149,185)	1.185	Trapezium
$1\to3\to4\to6$	(132,150,167,187)	0.980	Triangle
$1\to3\to5\to6$	(110,141,154,180)	1.129	Trapezium

Simulation Result using C Language:



Results and Discussions: One way to verify the solution obtained is to make an exhaustive comparison

- Shiang-Tai and Chiang [17] used Yager method for Fig. 4 and obtained $P_3 = 1 \to 2 \to 3 \to 5 \to 6$ as the shortest path. The solution is the same as that solved by the method of this paper. That is for the measures (centroid measure, alphacut measure/mean-standard deviation measure), the fuzzy shortest path remains the same namely $P_3 = 1 \to 2 \to 3 \to 5 \to 6$ but the fuzzy shortest path length differs namely 429.67, 287 respectively. From here we come to the conclusion that there exists no unique method for comparing fuzzy numbers and different methods may satisfy different desirable criteria.
- Okada and Soper [13] developed an algorithm based on the multiple labelling methods for a multicriteria shortest path to find a number of non-dominated paths. For Fig.4, two non-dominated paths $P_3 = 1 \to 2 \to 3 \to 5 \to 6$ with $L_3 = (103, 137, 149, 185)$ and $P_5 = 1 \to 3 \to 5 \to 6$ with $L_5 = (110, 141, 154, 180)$ are obtained, which include the solution solved by the method of this paper that is P_3. Although P_3 and P_5 are defined to be non-dominated by Okada and Soper [13], the ranking measures defined in this paper indicate that P_3 is dominated by P_5, in otherwords, path P_3 has the shortest distance. Hence we conclude that the solution derived from this study is better than that solved in [13].

Conclusion : Many researchers have focused on the FSPP in a network, since it is significant for various applications. In this paper, few defuzzification formulae and three different algorithms are developed for solving shortest path problem on a network with fuzzy edge weights. Simulation result is included to show the presented algorithms in detail. It can also be found that the ranking given to the paths can be helpful to the decision-makers as they make decision in choosing the best optimal path. Hence we conclude that the algorithms developed in this paper can be the simplest way to identify the shortest path in fuzzy environment.

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