MANY LIVES OF LATTICE SIGMA ALGEBRAS

J. PRAMADA

ABSTRACT: In this paper we define the concepts of lattice measurable set, lattice measure space and lattice sigma– finite measure are defined. Here we provide some basic elementary properties of lattice algebra, lattice sigma algebra and establishes many lives of lattice sigma algebras.

Keywords: lattice measurable set, lattice measure space, lattice sigma– finite measure.

Introduction: The fundamentals of measure theoretical concepts were firstly described by Halmos (1974). Later on, this concept was attempted by Royden (1981). The concept of measure of a lattice has been endeavored by Szasz (1963). The horizon of the measure of a lattice has been efforted by G. Szasz (1963) and signed lattice measure is originated by Tanaka (2009).

In section 2, plinth on Tanaka (2009), the fundamentals of lattice sigma algebra, lattice measure on a lattice sigma algebra were described. Further by means of Anil kumaretrl (2011) the concepts of lattice measurable set, lattice measure space and lattice σ – finite measure are defined. Here we provide some basic elementary properties of lattice algebra, lattice sigma algebra.

Priliminaries:

Lattice Algebras: Unless otherwise stated, X is the entire set and S is a lattice of any subsets of X.

Definition2.1. If S is a lattice and satisfies the following conditions, then it is called a lattice algebra. (i) $h \in S$, $h^c \in S$, (ii) For all $a, b \in S$, $a \vee b \in S$.

Result2.1. If $E_1, E_2 \in S$, $E_1 \wedge E_2 \in S$.

Proof. If $E_1 \in S$, by the definition 2.2.1., $E_1^c \in S$ again $E_2 \in S$, by definition 2.2.1., $E_2^c \in S$.Now $E_1^c, E_2^c \in S$, then $E_1^c \vee E_2^c \in S$.Which implies that $(E_1 \wedge E_2)^c \in S$.

Hence $E_1 \wedge E_2 \in S$.

Result2.2. If $E_1, E_2 \in S$, then $E_1 - E_2 \in S$.

Proof. Let $E_2 \in S$. By definition2.1. $E_2^c \in S$. Now $E_1, E_2^c \in S$. Which implies that $E_1 \wedge E_2^c \in S$ (by result 2.1.). Then, $E_1 \wedge E_2^c = E_1 - E_2 \in S$.

Theorem2.1. If S is lattice algebra of subsets of X then (i) $X \in S$ (ii) $\phi \in S$.

Proof.

1. Since S is nonempty, there exists $A \in S$. Hence $A^c \in S$ so $X = A \vee A^c \in S$.
2. Clearly $\phi = X^c \in S$.

Theorem2.2. Suppose that S is lattice algebra of subsets of X and that $A_i \in S$ for each i in a finite index set I.

(i) $\vee_{i \in I} A_i \in S$ (ii) $\wedge_{i \in I} A_i \in S$.

Proof. We prove this theorem by using induction on the number of elements in I. Let $x_1, x_2, \ldots\ldots x_n \in S$. Since $x_1, x_2 \in S$ and S is a lattice, we have that $x_1 \wedge x_2$, $x_1 \vee x_2 \in S$. Suppose the induction hypothesis that $x_1 \wedge x_2 \wedge \ldots\ldots x_{n-1}$, $x_1 \vee x_2 \vee \ldots\ldots \vee x_{n-1} \in S$. Since $x_1 \wedge x_2 \wedge \ldots\ldots x_{n-1} \in S$, $x_n \in S$ and S is a lattice, We have $x_1 \wedge x_2 \ldots\ldots \wedge x_{n-1} \wedge x_n \in S$. Since $x_1 \vee x_2 \vee \ldots\ldots \vee x_{n-1} \in S$, $x_n \in S$ and S is a lattice. We have that $x_1 \vee x_2 \vee \ldots\ldots \vee x_{n-1} \vee x_n \in S$.It is clear that $x_1 \wedge x_2 \ldots\ldots \wedge x_n$ is the $\inf\{x_1, x_2, \ldots\ldots x_n\}$ and $x_1 \vee x_2 \vee \ldots\ldots x_n$ is the $\sup\{x_1, x_2, \ldots\ldots x_n\}$.This theorem is true for all positive integers n.Therefore $\vee_{i \in I} A_i \in S$ and $\wedge_{i \in I} A_i \in S$.

Lattice σ-Algebras.

Unless otherwise stated, X is the entire set and S is a lattice of any subsets of X.

Definition2.2. If a lattice S satisfies the following conditions, then it is called a lattice σ - algebra;(1)For all $h \in S$, $h^c \in S$.(2)If $h_n \in S$ for n = 1, 2, 3 ....., then $\vee_{n=1}^{\infty} h_n \in S$.

Theorem2.3. S = {A < X / either A is countable or A^c is countable} is a lattice σ- algebra.

Proof. Part 1: To Prove that $X \in S$.We have $X^c = \phi$ is countable ($\phi$by convention) and by definition of lattice σ- algebra. $\phi^c = X \in S$.

Part 2: Let $A \in S$ we show that $A^c \in S$. Since $A \in S$ we have either A or A^c is countable.

Now if A is countable then A^c is the complement of a countable set and thus it is in S.

Part 3: Let $A_1, A_2, \ldots \in S$ we show that $\vee_{n=1}^{\infty} A_n \in S$.Assume that $A_1, A_2, ..$ are countable . Implies $\vee_{n=1}^{\infty} A_n$ is countable.Suppose that there exists, $n_0 \in N$ such that A_{n_0} is not countable.Implies $\left(\vee_{n=1}^{\infty} A_n\right)^c = \wedge_{n=1}^{\infty} A_n^c < A_{n_0}^c$ .But $\left(\vee_{n=1}^{\infty} A_n\right)^c$ is a sub set of a countable set and hence it is itself countable.

Definition2.3 Let R be a real number system and S be a lattice σ –algebra of X. If a mapping $\mu : S \longrightarrow R \vee \{\infty\}$ satisfies the following properties, then μ is called a lattice measure on the lattice σ -algebra S.(1) $\mu(\phi) = \mu(0) = 0$.(2)For all $h, g \in S$ such that $\mu(h), \mu(g) \geq 0$; $h \leq g \Longrightarrow \mu(h) \leq \mu(g)$.(3)For all $h, g \in S$, $\mu(h \vee g) + \mu(h \wedge g) = \mu$

(h) + μ(g).(4)If $h_n \in S$, $n \in N$ such that $h_1 \leq h_2 \leq \ldots \leq h_n \leq \ldots$, then $\mu(\vee_{n=1}^{\infty} h_n) = \lim \mu(h_n)$.

Let μ_1 and μ_2 be lattice measures defined on the same lattice σ -algebra S. If one of them is finite, then the set function $\mu(E) = \mu_1(E) - \mu_2(E)$, $E \in S$ is well defined and countably additive on S. However, it is necessarily nonnegative; it is called a signed lattice measure.

Definition2.4. Let X be a non-empty set and S be a lattice σ -algebra of subsets of X. Then ordered pair (X, S) is said to be lattice measurable space.

Definition2.5. A set A is said to be lattice measurable set or lattice measurable if A belongs to S.

Theorem2.4. If $A_i \in S$ for each i in a countable index set I, then $\wedge_{i\in I} A_i \in S$.

Proof. We prove this theorem by using theorem 2.2. If $A_i \in S$ for i∈I, then $A_i^c \in S$ for i∈I.

Therefore $\vee_{i\in I} A_i^c \in S$ .Hence $\wedge_{i\in I} A_i = (\vee_{i\in I} A_i^c)^c \in S$.

Theorem2.5. Suppose that S is a set and S is a finite lattice algebra of subsets of X. Then S is also a lattice σ-algebra.

Proof. We prove this theorem by using theorem 2.2.We just add a lot of empty sets, that is,

$A_1 \vee A_2 = A_1 \vee A_2 \vee \phi \vee \phi \vee \ldots\ldots\ldots$

Now we have an infinite sub sets.

Note2.5. However, there are lattice algebras that are not lattice σ-algebras.

The following theorem is an example of aforesaid note.

Theorem2.6. The collection of finite and co-finite subsets of set of natural number system N defined below is a lattice algebra of subsets of N, but not a lattice σ-algebra,

$F = \{A \subseteq N: A \text{ is finite or } A^c \text{ is finite}\}$.

Proof. We have $N \in F$, since $N^c = \phi$ is finite. If $A \in F$, then $A^c \in F$ by the symmetry of the definition lattice algebra. Suppose that $A, B \in F$. If A and B are both finite, then $A \vee B$ is finite.

If A^c or B^c is finite, then $(A \vee B)^c = A^c \wedge B^c$ is finite. In either case, $A \vee B \in F$. Thus F is lattice algebra of subsets of N. Let $A_n = \{2n\}$ for $n \in N$. Then A_n is finite. So $A_n \in F$ for each $n \in N$.

Let $E = \vee_{n=0}^{\infty} A_n$. Note that E and E^c are infinite. So $E \notin F$.

Thus F is not a lattice σ-algebra.

Note2.6. Let P(X) denotes the collection of all subsets of X, called the power set of X. Trivially, P(X) is the largest lattice σ-algebra of X, at the other extreme, the smallest lattice σ-algebra of X is the collection $\{\phi, X\}$.

Many Lives Of Lattice Sigma Algebras:

Definition3.1. The lattice σ-Algebra F of all sub sets of X lies between $\{\phi, X\} < F < P(X)$ is called a lie lattice σ-Algebra.

Example3.1. A partition of X is a collection of disjoint subsets of X whose union is all of X. For simplicity, consider a partition consisting of a finite number of sets $A_1, A_2, \ldots\ldots$ Thus

$$A_i \wedge A_j = \phi \text{ and } A_1 \vee A_2 \ldots. A_n = X$$

Then the collection F consisting of all unions of the sets A_j forms a lie lattice σ - algebra.

Note3.1. All non trivial lattice σ – algebras are lie lattice σ– algebras.

Theorem3.1. If F be a lie lattice σ - algebra of subsets of X then it has the following.

(1) $X \in F$

(2) If $A_1, A_2 \ldots\ldots A_n \in F$, then $A_1 \vee A_2 \vee \ldots\ldots.. A_n \in F$

(3) If $A_1, A_2 \ldots\ldots A_n \in F$, $A_1 \wedge A_2 \wedge \ldots\ldots. A_n \in F$

(4) If $A_1, A_2 \ldots\ldots$ is a countable collections of sets in F then $\wedge_{n=1}^{\infty} A_n \in F$

(5) If $A, B \in F$ then $A - B \in F$.

Proof. Part(1). Since $\phi \in F$ and $X = \phi^c$ it follows that $X \in F$.

Part(2). (ii) we have $A_1 \vee A_2 \ldots. A_n = A_1 \vee A_2 \ldots. A_n \vee \phi \vee \phi \vee \ldots\ldots \in F$ (by definition6.2.1.)

Part(3). follows by complementation:

$A_1 \wedge A_2 \ldots. A_n = (A_1^c \vee A_2^c \ldots\ldots A_n^c)^c$ which is in F because each $A_i^c \in F$ and, by (i), F is closed under finite unions.

Part (4). Follows by taking complements:

$\wedge_{n=1}^{\infty} A_n = \left[\vee_{n=1}^{\infty} A_n^c\right]^c$ which belongs to F because F is closed under complements and countable unions.

Part(5). $A - B = A \wedge B^c$ is in F, because $A, B^c \in F$.

Indiscrete lattice σ - algebra:

Definition3.2 Let B a non – empty collection of subsets of a set X. The smallest lattice σ - algebra containing all the sets of B is denoted by σ(B) and is called the indiscrete lattice σ- algebra generated by the collection B.

Note3.2. Any lattice σ - algebra containing the sets of B must contain all the sets of σ(B).

Theorem3.2. If G is any non – empty collection of lie lattice σ - algebras of subsets of X, then the meet $\wedge G$ is indiscrete lie lattice σ - algebra of subsets of X.

That is $\wedge G = \{A < X \mid A \in F \text{ for every } F \in G\}$ consists of all sets A which belong to each lie lattice σ - algebra F of G.

Proof. Proof is very clear.

Note3.3. Given a collection B of subsets of X, let G_B be the collection of all lattice σ - algebras including containing all the sets of B. Note that $P(X) \in G_B$ and so G_B is non empty. Then $\wedge G_B$ is a lie lattice σ - algebra, contains all the sets of B, and is minimal among such lie lattice σ - algebras. Minimally here means if F is a lie lattice σ - algebra such that $B < F$ then $\wedge G_B < F$ thus $\wedge G_B$ is the lie lattice σ - algebra. This lie lattice σ- algebra is a indiscrete lie lattice σ - algebra.

Conclusion: This paper verifies some basic elementary properties of lattice algebras and lattice sigma algebras. Finally we proved many lives of lattice sigma algebras.

References:

1. Halmos. P.R., Measure Theory (Springer, New York, 1974).
2. Royden. H.L., Real Analysis, 3rd ed., Macmillan Publishing, New York, 1981.
3. Szasz Gabor, Introduction to lattice theory, academic press, New York and London1963.
4. Tanaka. J, Hahn Decomposition Theorem of Signed Lattice Measure, arXiv:0906.0147Vol1 [Math.CA] 31,May 2009.

* * *

J. Pramada
Bharat Institute of Engineering & Technology/Hyderabad/
A.P. India/pramadadaita@yahoo.co.in