ON VARIOUS DECOMPOSITIONS OF A SIGNED LATTICE MEASURE

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Abstract: This paper describes that the concepts of lattice measurable space, lattice measure space, lattice finite measure and lattice finite measure space and the perceptions of lattice σ – finite measure, lattice σ - finite measure space, absolutely continuous lattice measure and also establishes the various decompositions of a signed lattice measures.

Keywords: lattice measurable space, lattice σ - finite measure space, absolutely continuous lattice measure and signed lattice measures.

Introduction: In 2005 Mona Khare and BhawnaSingh [2] introduced the notation of a signed measure arises if a measure is allowed to take both positive and negative values. A set that is both positive and negative with respect to a signed measure is termed as a null set. They generalized some concepts in measure theory by means of class of null set. In 1963, GabourSzasz [3] Introduction To Lattice Theory explained much about sublattice concept like if P(M) denotes the set of all sub sets of M including the void set is a lattice with respect to the operations of set union and intersection. This lattice is called sublattice of M. He also explained that the set union and the set intersection of any two closed sets of a plane are likewise closed. Hence, The closed sets of a plane form a lattice with respect to these set operations. For similar reasons, the corresponding statement holds for all open sets of a plane also every single element sub set of a partially ordered set is an interval and this interval is a sub lattice, also this single element sub set {a} of a lattice L can be recorded as a zero length chain or as an interval [a, a], so every {a} is a sublattice of L as well as measurable [m(a) = 0].

In section2, we defined the concepts of lattice measurable space, lattice measure space, lattice finite measure and lattice finite measure space and the perceptions of lattice σ – finite measure, lattice σ - finite measure space, absolutely continuous lattice measure and also establish the various decompositions of a signed lattice measures.

Various Decompositions Of A Signed Lattice Measure:

Definition2.0. Let X is the entire set and L is a lattice of any subsets of X. If a lattice L satisfies the following conditions, then it is called a lattice σ-Algebra;

1. $\forall h \in L$, $h^c \in L$.
2. If $h_n \in L$ for n = 1, 2, 3 ....., then $\bigvee_{n=1}^{\infty} h_n \in L$.

We denote ß, as the lattice σ-Algebra generated by L.

Definition2.1. By a signed lattice measure on the lattice measurable space (X, ß) we mean a mapping ν: ß $\longrightarrow$ R $\vee$ {∞}orR $\vee$ {-∞}, satisfying the following properties

(1) $\nu(\phi) = \nu(0) = 0$.

(2)(i) For all h, g ∈ ß such that $\nu(h), \nu(g) \geq 0; h \leq g \Rightarrow \nu(h) \leq \nu(g)$.

(ii) For all h, g ∈ ß such that $\nu(h), \nu(g) \leq 0; h \leq g \Rightarrow \nu(h) \leq \nu(g)$.

(3)For all h, g ∈ ß, $\nu(h \vee g) + \nu(h \wedge g) = \nu(h) + \nu(g)$.

(4)If h_n<ß, n ∈ N such that $h_1 \leq h_2 \leq ... \leq h_n \leq ....$, then $\nu(\bigvee_{n=1}^{\infty} h_n) = \lim \nu(h_n)$.

Defintition2.2. The ordered pair (X, ß) is said to be lattice measurable space.

Definition2.3. A is called a positive lattice if A is lattice measurable and for any lattice measurable set E in A, $\nu(E) \geq 0$; Similarly, B is called a negative lattice if B is a lattice measurable and for any lattice measurable set E in B, $\nu(E) \leq 0$.

Definition2.4. (**Positive and Negative parts of ν**)**:** Let (X, ß) be a lattice measurable space and ν be a signed lattice measure defined on ß. If we define ν^+ and ν^- on for all E ∈ ß such that $\nu^+(E) = \nu(E \wedge A)$ and $\nu^-(E) = -\nu(E \wedge B)$ with $\nu = \nu^+ - \nu^-$. Then ν^+ and ν^- are called the positive and the negative parts of a signed lattice measure respectively. Where {A, B} be a lattice Hann decomposition on X.

Definition2.5. (Mutually singular lattice measures): Let (X, ß) be a lattice measurable space and ν be a signed lattice measure defined on ß. If we define two measures ν_1 and ν_2 are said to be mutually singular lattice measures with each other to be denoted by $\nu_1 \perp \nu_2$.That is if there exists two lattice measurable sets A and B such that $A \vee B = X$ and $A \wedge B = \phi$ and $\nu_1(A) = \nu_2(B) = 0$.

Definition2.6. The lattice measurable space (X, ß) together with a lattice measure μ is called a lattice measure space and it is denoted by (X, ß, μ).

Definition2.7. Let (X, ß, μ) be a lattice measure space. If μ(X) is finite, then μ is called lattice finite measure.

Definition2.8. If μ is a lattice finite measure then (X, ß, μ) is called a lattice finite measure space.

Definition2.9. Let (X, ß, μ) be a lattice measure space. If there exists a sequence of lattices measurable sets { x_n } such that

$X = \bigvee_{n=1}^{\infty} x_n$ (ii) $\mu(x_n)$ is finite.

Then μ is called a lattice σ – finite measure.

Definition2.10. If μ be a lattice σ – finite measure, then (X, ß, μ) is called lattice σ – finite measure space.

Definition2.11.Absolutely continuous lattice measure: Let (X, ß, μ) be a lattice measure space and ν be a lattice measure defined on ß for each sublattice A of X, whenever μ(A) = 0 implies ν(A) = 0. Then ν is said to be absolutely continuous lattice measure with respect to μ and it is denoted by $\nu << \mu$.

By using the above concepts one can establish the following theorems.

Theorem2.1. :

Lattice Hann-Decomposition: Let (X, ß) be a lattice measurable space and ν be a signed lattice measure defined on ß. Then there exists a positive lattice A and a negative lattice B such that $A \vee B = X$ and $A \wedge B = \phi$.

Theorem2.2.

Jordan–Decomposition On Signed Lattice Measure: Let (X, ß) be a lattice measurable space and ν be a signed lattice measure defined on ß. Then there exists a unique pair of mutually singular lattice measures ν^+ and ν^- with $\nu = \nu^+ - \nu^-$.

Theorem2.3. Radon – Nikodym Theorem On Signed Lattice Measure: Suppose (X, ß, μ) be a lattice σ-finite measure space and ν be a lattice measure defined on ß which is absolutely continuous with respect to μ. Then there exists, a non-negative lattice measurable function f such that for all E in ß we have $\nu(E) = \int_E f\, d\mu$. More over the lattice measure f is unique in this sense that if g is also a non-negative lattice measurable function such that $\nu(E) = \int_E g\, d\mu$, E ∈ ß then f = g almost everywhere on X with respect to μ.

Theorem2.4.

Lebesgue Decomposition Of A Signed Lattice Measure:

Suppose (X, ß,μ) be a lattice σ - finite measure space and ν be a lattice σ-finite measure defined on ß. Then we find

(1) A lattice measure ν_0 mutually singular lattice measure with respect to μ and

(2) A lattice measure ν_1 absolutely continuous lattice measure with respect to μ such that

(3) $\nu = \nu_0 + \nu_1$ more over ν_0 and ν_1 are unique.

Proof. Let (X, ß,μ) be a lattice σ-finite measure space and ν be a lattice σ-finite measure defined on ß. Now μ and ν are lattice σ - finite measures, therefore λ = μ + ν is also a lattice σ - finite measure. Also evidently, μ<<λ and ν<<λ. Therefore by (Radan - Nikodym Theorem) there exists two non- negative lattice measurable functions f and g on X such that for every E ∈ ß,

We have $\mu(E) = \int_E f\, d\lambda$--------------- (1) and

$\nu(E) = \int_E f\, d\lambda$---------- (2)

Now f is a non-negative lattice measurable function defined on X. Define two sets A and B such that A = {x ∈ X / f(x) > 0} and B = {x ∈ X / f(x) = 0}. Cleary $A \vee B = X$ and $A \wedge B = \phi$

from (1) $\mu(B) = \int_B f\, d\lambda = 0$. Therefore μ(B) = 0

Case (1). Define a measure ν_0 such for every E ∈ ß, we have $\nu_0(E) = \nu(E \wedge B)$.

Now $\nu_0(A) = \nu(A \wedge B) = \nu(\phi)$.Therefore there exists two lattice measurable sets A and B such that $A \vee B = X$ and $A \wedge B = \phi$ also $\nu_0(A) = \nu_0(B) = 0$. Therefore ν_0 and μ are mutually singular lattice measures.

Case (2). Define a lattice measure ν_1 such for every E ∈ ß, we have $\nu_1(E) = \nu(E \wedge A)$.

Let E be a lattice measurable set of μ measure zero.

Now μ(E) = 0. Implies ∫ f dλ = 0 (since (1))

Implies λ(E) = 0. Implies $\lambda(E \wedge A) = 0$.

Implies $\mu(E \wedge A) + \nu(E \wedge A) = 0$ (sinceλ = μ+ν)

Implies $0 + \nu(E \wedge A) = 0$ (since μ(E) = 0).

Implies $\nu(E \wedge A) = 0 = \nu_1(E) = 0$.

Therefore $\mu(E) = \nu_1(E) = 0$. Henceν_1 absolutely continuous lattice measure with respect to μ

Case (3). For every E ∈ ß, we have

$E = (E \wedge A) \vee (E \wedge B)$.

Implies $\nu(E) = \nu(E \wedge A) + \nu(E \wedge B)$

Implies $\nu(E) = \nu_0(E) + \nu_1(E)$. Implies $\nu = \nu_0 + \nu_1$.

Uniqueness: Suppose there exists two pair of lattice measures ν_0, ν_1 and ν_0^1, ν_1^1

Such that $\nu = \nu_0 + \nu_1$----- (3)

and $\nu = \nu_0^1 + \nu_1^1$------- (4).

Let {A, B} and {A^1, B^1} be two pair of lattice measurable sets as explained above and

$A \vee B = X$, $A \wedge B = \phi$ and $A^1 \vee B^1 = X$, $A^1 \wedge B^1 = \phi$

. $\mu(B) = \mu(B^1) = \nu_0(A) = \nu_0(A^1) = 0$ -- (5),

For every E ∈ ß, we have $E = (E \wedge A \wedge A^1) \vee (E \wedge A \wedge B) \vee (E \wedge A^1 \wedge B) \vee (E \wedge B \wedge B^1)$… (6).

Since $\mu(B) = \mu(B^1) = 0$.The μ lattice measure of last three terms of (6) are zero. But $\nu_1 << \mu$

Implies the ν_1 lattice measure of last three terms of (6) are zero. (3) – (4) gives $(\nu_0 - \nu_0^1) + (\nu_1 - \nu_1^1) = 0$. Implies $(\nu_0 - \nu_0^1) = (\nu_1^1 - \nu_1)$ ------ (7). In view of the above argument for every E ∈ ß,

$(\nu_1^1 - \nu_1)(E) = (\nu_1^1 - \nu_1)(E \wedge A \wedge A^1)$. Implies $(\nu_1^1 - \nu_1)(E) = (\nu_0 - \nu_0^1)(E \wedge A \wedge A^1) = 0$ (since (5)).

This is true for every E ∈ ß,that is $\nu_1^1 - \nu_1 = 0$. Implies $\nu_1^1 = \nu_1$.Therefore ν_1 is unique lattice measures.

Now from (7) $\nu_0 - \nu_0^1 = 0$. Implies $\nu_0 = \nu_0^1$. Implies ν_0 is unique lattice measure.

Therefore there exists a unique pair of lattice measures ν_0 and ν_1 such $\nu = \nu_0 + \nu_1$.

Conclusion: In this paper defined the concepts of lattice measurable space, lattice measure space, lattice finite measure and lattice finite measure space and the perceptions of lattice σ – finite measure, lattice σ - finite measure space, absolutely continuous lattice measure and also established the various decompositions of a signed lattice measures.

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