DISCRETE GROUP WITH THE TWISTED RAPID DECAY PROPERTY

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Abstract: Property RD provide estimates for the operator norm of those functions (in the left-regular representation) in terms of the Sobolev norm. We had studied the twisted property RD and will attempt to answer the question of existence of discrete group that satisfies twisted RD but that the ordinary RD. We use the connection between property RD and twisted Rapid Decay Property to show the some groups of twisted Rapid Decay Property. We show that the alternative characterisations of twisted Rapid Decay Property. In this paper we describe that the analytic properties of the twisted property RD property pass to direct products.

Keywords: Rapid Decay Property, Uniform Roe algebras, Twisted Rapid Decay property.

Introduction: The reader is referred to Jolissaint [4], Kannan [5], [6], [7] and Chatterji [2, 1] for details of the property (RD). The rapid decay property for groups, generalizes Haagerup's [3] inequality for free groups and so for example of free groups have property RD. This property RD for groups has deep implications for the analytical, topological and geometric aspects of groups. Jolissaint proved in his thesis that groups of polynomial growth and classical hyperbolic groups have property RD, and the only amenable discrete groups that have property RD are groups of polynomial growth. He also showed that many groups, for instance SL3(Z), do not have the Rapid Decay property [4]. Examples of RD groups include group acting on CAT (0)-cube complexes [2], hyperbolic groups of Gromov, Coxeter groups [2], and torus knot groups [4].

In Section 2, we first recall the property RD and length functions and in section 3, we study twisted property RD. We show that the alternative characterisations of twisted Rapid Decay Property.

Proposition: Let σ be a multiplier on G and ℓ be length function on G. Then the following are equivalent: G has $\sigma -$ twisted Rapid Decay property (with respect to the length ℓ) if and only if

$\|f *_\sigma g \ *_\sigma h\| \leq P(r) \|f\|_{\ell^2(G)} \|g\|_{\ell^2(G)} \|h\|_{\ell^2(G)}$

We had studied the twisted property RD and will attempt to answer the question of existence of a discrete group that satisfies twisted RD but that the ordinary RD. We provide a simple method to establish the connection between property RD and twisted Rapid Decay Property. In particular, there is the following implication for groups:

Property (RD) $\Rightarrow$ twisted property RD.

We then use this to show the following groups have twisted property RD: CAT(0)-cube complexes, hyperbolic groups of Gromov, Coxeter groups, torus knot groups, Crystallographic groups, the Discrete Heisenberg group, the abelian free group of rank $R > 1$ and the non-abelian free group of rank $R > 2$. $SL_3(Z)$ does not have Property RD but which has twisted Rapid Decay property. We describe that the analytic properties of twisted property RD pass to direct products.

Proposition: The direct product of two discrete groups with the twisted property RD has the twisted property RD.

Rapid Decay Property: We will explain the basic notations related to property RD for discrete groups in [8].

Definition 2.1: Let G be a locally compact group. A length function on G is a map $\ell: G \to R$ taking values in the non-negative reals which satisfies the following conditions:

- $\ell(1) = 0$, where 1 is the identity element of the group;
- For every g in G, $\ell(g) = \ell(g^{-1})$;
- For every g, h in G,
- $\ell(gh) \leq \ell(g) + \ell(h)$.

A group equipped with a length function becomes a metric space with the left-invariant metric

$$d(\gamma, \mu) = \ell(\gamma^{-1}, \mu.).$$

Definition 2.2: Let map ℓ be a length function on G. We define a Sobolev norm on the group ring of C [G] as follows:

- If s in $\mathbb{R}$, the Sobolev space of order s is the set $H_\ell^s(G)$ of functions ξ on G such that $\xi(1 + \ell)^s$ belongs to $\ell^2(G)$.
- For any length function ℓ and positive real number s, we define a Sobolev norm on the group ring C[G] by

$$\|f\|_{\ell,s} = \sqrt{\sum_{\gamma \in G} |f(\gamma)|^2 (1 + \ell(\gamma))^{2s}}$$

We are now ready to define property RD. The following definition is due to Jolissaint [8].

Definition 2.3: Let ℓ be a length function on a discrete group G. We say that G has the property (RD) with respect to the length function ℓ if there exist $C \geq 0$ and $s > 0$ such that, for all $f \in C[G]$,

$$\|f\|_* \leq C \|f\|_{\ell,s}$$

where $\|f\|_*$ denotes the operator norm of f acting by left convolution on $\ell^2(G)$.

Definition 2.4: We say that a discrete group G has polynomial growth with respect to a length function ℓ if there exists a polynomial Psuch that the cardinality of the ball of radius r (denoted by $|B(e, r)|$) is bounded by $P(r)$.

Example 2.5: Let G be a discrete group endowed with a length function ℓ with respect to which G is of polynomial growth. Then G has property RD with respect to ℓ .

Author shows that the Crystallographic groups, the

Discrete Heisenberg group, have property RD.

3. Twisted Rapid Decay: Let σ is a multiplier on G discrete group. A 2-cocycles over G (time the normalized U(1) -valued- 2 - cocycle on G) is a linear map $\sigma : G \times G \rightarrow U(1)$ satisfying the following identity for all $\gamma, \mu, \delta \in G$:

1. $\sigma(\gamma,\mu)\sigma(\gamma\mu,\delta) = \sigma(\gamma,\mu\delta)\sigma(\mu,\delta)$.
2. $\sigma(\gamma,1) = \sigma(1,\gamma)$.

Recall that the Dixmier-Douady invariant of a multiplier σ is the cohomology class

$\delta(\sigma) \in H_3(\gamma, Z)$,.The image of $[\sigma]$ obtained under the map δ arising in the long exact in cohomology derived from the short exact sequence of coefficients

$$0 \rightarrow Z \rightarrow R \rightarrow U(1) \rightarrow 0.$$

Definition 3.1 Let A be $C^* -$ algebra, we denote by $C(G, A, \gamma)$ as finite sums $\sum a_\gamma T_\gamma$

Where $a_\gamma \in A$, T_γ is a letter satisfying.

1. $T_\gamma T_\mu = \sigma(\gamma,\mu)T_{\gamma\mu}$.
2. $T_\gamma a T_\gamma{}^* = \alpha\gamma(a)$.
3. $T_\gamma{}^* = \gamma(\gamma, \gamma - 1)T_\gamma{}^{-1}$.

Given a Banach norm $\| \quad \|_B$ on $C(G, A, \gamma)$, we denote by $B(G, A, \gamma)$ the completion of $C(G, A, \gamma)$ with respect to the norm $\| \quad \|_B$.

In case where $A = C$ (with trivial G–action) we simply write $C(G, \sigma)$. We often represent it as the C− subalgebra of $B(\ell_2(G))$ generated by $\{T_\gamma : \gamma \in G\}$, where $\gamma \in G$,

$$T_\gamma : \ell_2(G) \longrightarrow \ell_2(G)$$
$$\delta_\mu \longmapsto \sigma(\gamma,\mu)\delta_{\gamma\mu} .$$

So that an element in $C(G, \sigma)$ is a finite linear combination of the operators T_γ and the convolution reads (for $\gamma, \mu \in G$)

$$T_\gamma * \sigma T_\mu = \sigma(\gamma,\mu)T_{\gamma\mu}.$$

We shall consider several completions of $C(G, \sigma)$. The ℓ_1− completion (given by the norm

$$\left\|\sum_{\gamma \in G} a_\gamma T_\gamma\right\|_1 = \left\|\sum_{\gamma \in G} a_\gamma\right\|$$

yields the ℓ_1− twisted Banach algebra denoted by $\ell_1(G, \sigma)$, which is the completion of $C(G, \sigma)$ with respect to this $\ell_1 -$ norm. It is straightforward computation to show that it is indeed a Banach algebra, contained in $B(\ell_2(G))$. Next we shall consider the operator norm given by

$$\|f\|_{op} = \sup\{\|f(x)\|_2 : \|x\| = 1\}$$

and the completion of $C(G, \sigma)$ with respect to this norm is the twisted reduced C^*– algebra $C_r^*(G, \sigma)$. For any length function ℓ and positive real number s, we define the s-weighted $-\ell_2$ norm is defined by:

$$\left\|\sum_{\gamma \in G} a_\gamma T_\gamma\right\|_s = \sqrt{\sum_{\gamma \in G} |a_\gamma|^2 ((1 + \ell(\gamma))^{2s})}$$

and the s− Solbolev space is the completion of $C(G, \sigma)$ with respect to this norm , denoted by $H_\ell^\infty(G, \sigma)$. If the length function is chosen to be word length with respect to a finite set of generators for G, then write $H_s(G, \sigma)$. The space of rapidly decreasing functions is given by

$$H_\ell^\infty(G, \sigma) = \bigcap_{s>0} H_\ell^s(G, \sigma)$$

Let Gbe finitely generated group, endowed with a length function ℓ and σ is a multiplier on G. If the group G has σ− twisted Rapid decay property if

$$H_\ell^\infty(G, \sigma) \subseteq C_r^*(G, \sigma)$$

We say that the group G has the rapid decay property (with respect to the length l), if it has the σ− twisted Rapid decay property (with respect to the length ℓ for the constant multiplier 1.

Definition 3.2 We say that a group G has property σ− RD if there exists a length function ℓ with respect to which G has the σ− twisted Rapid decay property.

In the context of non-commutative geometry, the reduced C^*- algebra $C_r^*(G, \sigma)$ represents the space of continuous functions on a non-commutative manifold, and $H_\ell^\infty(G, \sigma)$ the smooth functions on the same non-commutative manifold. This comes from abelian case, where using Fourier transforms, one easily sees that $C_r^*(\mathbf{Z}^n) \cong C(\mathbf{T}^n)$ and $H_\ell^\infty(\mathbf{Z}^n) \cong C(\mathbf{T}^n)$ (for the word length associated to the generating set $S = \{(\pm 1, 0, \dots), \dots, (0, \dots, \pm 1)\}$ of $\mathbf{Z}^n$). The (σ −twisted) Rapid Decay property can be rephrased as the desirable property that every smooth function on the non-commutative manifold is also a continuous function. The following Proposition due to Chatterji.

Theorem 3.3 Let σ be a multiplier on G and ℓ be length function on G. Then the following are equivalent:

1. G has σ− twisted Rapid Decay property (with respect to the length ℓ).
2. $\exists\, C, s > 0$ such that for any $f \in C(G, \sigma)$, $\|f\|_{op} \leq C\|f\|_s$,
3. There exists a polynomial P such that for any $f \in C(G, \sigma)$ and f supported in a ball of radius r,
$$\|f\|_{op} \leq P(r)\|f\|_{\ell_2(G)}$$
4. There exists a polynomial P such that for any$f, g \in C(G, \sigma)$ and f supported in a ball of radius r
$$\|f * g\|_{op} \leq P(r)\|f\|_{\ell_2(G)}\|g\|_{\ell_2(G)}$$

We denote by $\mathbb{R}_+G$ the subset $C[G]$ consisting of functions with target in $\mathbb{R}_+$.

We show that the alternative characterisations of twisted Rapid Decay Property.

Main Results:

We show that the following proposition:

Proposition 4.1: Let σ be a multiplier on G and ℓ be length function on G.. Then the following are equivalent: G has σ − twisted Rapid Decay property (with respect to the length ℓ) if and only if

$$\|f *_\sigma g *_\sigma h\| \leq P(r) \|f\|_{\ell^2(G)}\|g\|_{\ell^2(G)}\|h\|_{\ell^2(G)}$$

Proof: Direct part by above Proposition 3.3. On the hand: for$f \in \mathbb{R}_+G$,

$$\|f\|_{op} = \sup\left\{\frac{\|f * g\|_2}{\|g\|_2} : g \in \ell^2(G)\right\}.$$

Since $\langle f * g|h\rangle \leq \|f * g\|_2\|h\|_2$.

With equality $f * g = h$.

It follows that $\|f * g\|_2 = \sup\left\{\frac{\langle f*g|h\rangle}{\|h\|_2} : h \in \ell^2(G)\right\}$.

We get

$\|f\|_{op} = \sup\left\{\frac{\langle f*g|h\rangle}{\|g\|_2\|h\|_2} : g,h \in \ell^2(G)\right\}.$

Since $\langle f*g|h\rangle = \sum_{\gamma\in G}(f*g)(\gamma)\overline{h(\gamma)}$

$$= \sum_{\gamma\in G}(f*g)(\gamma)\, h^*(\gamma^{-1})$$

$$= f*g*h^*(e).$$

where $h^*(\gamma^{-1}) = h^*(\gamma)$.

We obtain

$$\|f\|_{op} = \sup\left\{\frac{f*g*h^*(e)}{\|g\|_2\|h\|_2} : g,h \in \ell^2(G)\right\}$$

$$= \sup\left\{\frac{f*g*h(e)}{\|g\|_2\|h\|_2}\ g,h \in \ell^2(G)\right\}$$

$$\leq \sup\left\{\frac{P(r)\,\|f\|_{\ell^2(G)}\|g\|_{\ell^2(G)}\|h\|_{\ell^2(G)}}{\|g\|_2\|h\|_2} : g,h \in \ell^2(G)\right\}.$$

Then $\|f\|_{op} \leq P(r)\|f\|_2$.

Then G has σ− twisted Rapid Decay.

The following Proposition due to Chatterji. In particular, there is the following implication for groups:

Proposition 4.2 Let G be finitely generated group. If G has property RD then G has twisted property RD.

We then use this to show the following groups have twisted property RD:

- CAT(0)-cube complexes. [2]
- Hyperbolic groups of Gromov .[2]
- Coxeter groups. [2]
- Torus knot groups. [4]
- Crystallographic groups. [5]
- The Discrete Heisenberg group. [5]
- The abelian free group of rank $R > 1$.[5]
- The non-abelian free group of rank $R > 2$.[5]

Jolissaint proved in his thesis that groups of polynomial growth and classical hyperbolic groups have property RD, and the only amenable discrete groups that have property RD are groups of polynomial growth. He also showed that many groups, for instance $SL3(Z)$, do not have the Rapid Decay property [4]. $SL3(Z)$ does not have Property RD but which has twisted Rapid Decay property.

We describe that the analytic properties of Twisted property RD property pass to direct products.

Proposition 4.3 The direct product of two discrete groups with the twisted property RD has the twisted property RD.

Proof: Let H and K are two discrete groups with the twisted property RD.

Then $H_\ell^\infty(H,\sigma) \subseteq C_r^*(H,\sigma)$

and $H_\ell^\infty(K,\sigma) \subseteq C_r^*(K,\sigma)$

We have $H_\ell^\infty(H\times K,\sigma) = H_\ell^\infty(H,\sigma) \times H_r^*(K,\sigma)$

$\subseteq C_r^*(H,\sigma) \times C_r^*(K,\sigma)$

$= C_r^*(H\times K,\sigma)$

So G has the twisted property RD.

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