RECTANGULAR NEAR – IDEMPOTENT SEMIGROUP

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Abstract: In this paper we define a rectangular near- idempotent semigroup, a left (right) zero near – idempotent semigroup and a near – null semigroup. We obtain a decomposition of a rectangular near idempotent semigroup into left (right) zero near – idempotent semigroup and a left (right) zero near- idempotent semigroup into near- null semigroups.

Keywords: Left (right) zero near idempotent semigroup, near – null semigroup., rectangular near – idempotent semigroup

Introduction: In a previous paper we introduced a near–idempotent semigroup [3] and studied its structure through relations λ, ρ and δ similar to Green's relations[1].We also obtained a semilattice decomposition of a near – idempotent semigroup into what we call rectangular near- idempotent semigroups. Each of these rectangular near- idempotent semigroups is indeed, a δ – class.

In this paper we study the rectangular near- idempotent semigroups further and decompose it into smaller semigroups which we call left(right) zero near idempotent semigroups. The left zero near – idempotent semigroups are in fact, the λ – classes and the right zero near – idempotent semigroups are the ρ – classes lying in every given δ- class.

We recall that a near – idempotent semigroup is a semigroup S whose elements satisfy the identity $xyz = xy^2z$ for all x, y, z in S.

We now define a rectangular near – idempotent semigroup as follows

Rectangular Near- idempotent:
Semigroup:

Definition A semigroup R is called a rectangular near idempotent semigroup if R is a near – idempotent semigroup and its elements satisfy the identity

$xyzyw = xyw$ for all $x, y, z, w \in R$.

We recall the definition of the relation λ and ρ on a near – idempotent semigroup in the following

Definition [3] If S is a near – idempotent semigroup and $a, b \in S$, then $a \lambda b$ if $xaby = xay$ and $xbay = xby$ for all $x, y \in S$.

Dually $a \rho b$ if $xaby = xby$ and $xbay = xay$ for all $x, y \in S$.

λ and ρ are left and right congruences on S respectively [3].

We now define a left(right) zero near – idempotent semigroup.

We further obtain a decomposition of a rectangular near idempotent semigroup into left(right) zero near – idempotent semigroup.

Left (Right) Zero Near – Idempotent Semigroup

Definition A semigroup B is called a left(right) zero near – idempotent semigroup if $xyzw = xyw$ ($xyzw = xzw$) for all x, y, z, w in B.

Lemma Let S be a near – idempotent semigroup. Then for $a \in S$, λ_a is left (right) zero near – idempotent semigroup

Proof: Let S be a near – idempotent semigroup Consider the relation λ on S.

For $a, b \in R$

$a \lambda b$ if and only if $xaby = xay$ and $xbay = xby$ for all $x, y \in S$.

Consider an equivalence class λ_a

Where $a \in S$.

We claim that λ_a is a left zero near – idempotent semigroup.

Let $u,v \in \lambda_a$

$a \lambda u$ and $a \lambda v$

Also by transitivity $u \lambda v$

For all $x, y \in S$

$x\,uv\,a\,y = xu.\,va\,y = xuvy$ and

$xa\,uv\,y = xau.v\,y = xavy = xay$

Hence $uv \lambda a$, so that $uv \in \lambda_a$

Thus λ_a is a subsemigroup of S.

Also $xuvy = xuy$ and $xvuy = xvy$ for all x, y in S, and have for all x, y in λ_a.

Thus for $x, u, v, y \in \lambda_a$

$xuvy = xuy$, by which λ_a is a left(right) zero near – idempotent semigroup.

Lemma: Let R be a rectangular near – idempotent semigroup. Then for $a, b \in R$, $\lambda_a\lambda_b \subset \lambda_b$.

Proof: Let $u \in \lambda_a$ and $v \in \lambda_b$.

Then $xuay = xuy$ and $xauy = xay$

$xvby = xvy$ and $xbvy = xby$

$xuv\,b\,y = xu\,vb\,y = xuvy$

$xb\,uvy = xbv\,uv\,y = xb\,vuvy = x\,bv\,y = xby$ ($u,v \in R$ and hence $xvuvy = xvy$)

Thus $uv \in \lambda_b$

i.e, $\lambda_a\lambda_b \subset \lambda_b$

Thus R is right zero semigroup of left zero near-idempotent semigroup.

Dually, we can prove that every ρ – class in a near – idempotent semigroup is a right zero near- idempotent semigroup and every rectangular near – idempotent semigroup is a left zero near- idempotent semigroup of right zero near- idempotent semigroups.

$(\rho_a \rho_b \subset \rho_a)$

Thus we have

Theorem: Every near – idempotent semigroup is a semilattice of rectangular near – idempotent semigroup each of which, in turn, is a left (right) zero near - idempotent semigroup of right(left) zero near- idempotent semigroup.

In what follows we define the relation ξ on a near – idempotent semigroup S.

Definition: Let S be a near – idempotent semigroup. Let $a, b \in S$. We say that $a \xi b$ if and only if $a \lambda b$ and $a \rho b$.

In other words $\xi = \lambda \cap \rho$.

Lemma: Let S be a near – idempotent semigroup. let $a,b \in S$. Then $a \xi b$ if and only if $xay = xby$ for all $x, y \in S$.

Proof Let $a \xi b$

Then $a \lambda b$ and $a \rho b$.

Hence for all $x, y \in S$.

$xaby = xay$; $xbay = xby$

and $xaby = xby$; $xbay = xay$

From the above equation it is clear that

$xay = xby$ for all x, y in S.

Conversely suppose that $xay = xby$ for all $x,y \in S$.

For all $x, y \in S$

$xaby = xbby = xb^2y = xby$

and $xbay = xaay = xa^2y = xay$ so that $a \rho b$. Also for all $x, y \in S$,

$xaby = xaay = xa^2y = xay$

and $xbay = xbby = xb^2y = xby$

so that $a \lambda b$

Thus $a (\lambda \cap \rho) b$, i.e, $a \xi b$.

Lemma: ξ is an equivalence relation on S.

Proof: Since $xay = xay$ for all $x, y \in S$

we have $a \xi a$

Hence ξ is reflexive.

Also $a \xi b \Rightarrow xay = xby$ for all $x, y \in S$,

which in turn implies $b \xi a$

Hence ξ is symmetric.

$a \xi b$ and $b \xi c \Rightarrow xay = xby$ for all $x, y \in S$ and

$xby = xcy$ for all $x, y \in S$

$\Rightarrow xay = xcy$ for all $x, y \in S$

$\Rightarrow a \xi c$.

Hence ξ is transitive.

Thus ξ is an equivalence relation on S.

Lemma:ξ is a congruence relation on the near – idempotent semigroup S.

Proof : Suppose $a \xi b$ and $c \xi d$.

For all x,y in S, $xay = xby$ and $xcy = xdy$

$xacy = x b cy = xbdy$, for all x, y in S.

$a \xi b$ and $c \xi d$ implies that $ac \xi bd$,

so that ξ is a congruence relation on S.

Near Null Semigroup:

We now define a near – null semigroup

Definition: Let S be a near – idempotent semigroup. it is called a near – null semigroup if $xay = xby$ for all $x, a, b, y \in S$.

Lemma: Let S be a near – idempotent semigroup. Let $a \in S$, then every ξ – class is a near null semigroup.

Proof Define ξ on S.

Let $a \in S$.

Let $u, v \in \xi_a$

$xuy = xay = xvy$ for all $x,y \in S$.

For all $x,y \in S$

$xuvy = x u. vy = xavy = xa.vy = xa.ay = x a^2y = xay$

Then $uv \in \xi_a$

so that ξ_a is a subsemigroup of S.

Also $xuy = xvy$ for all $x, y \in S$.

Hence $xuy = xvy$ for all $x, y \in \xi_a$ also.

In other words if $x, y z, w \in \xi_a$.

$xyw = xzw$

Hence ξ_a is a near null semigroup.

Let $u \in \xi_a$ and $v \in \xi_b$

$xuy = xay$ and $xvy = xby$

$xuvy = xavy = xaby$ so that $uv \in \xi_{ab}$

Hence $\xi_a \xi_b \subset \xi_{ab}$

The following lemma is immediate

Lemma Let S be a left zero near- idempotent semigroup and let $a, b \in S$.

Then $\xi_a \xi_b \subset \xi_{ab}$

Proof $xaby = xay$ and $xbay = xby$

Let $u \in \xi_a$ and $v \in \xi_b$

By the last lemma

$xuvy = xaby = xay$ so that $uv \in \xi_a$

Hence S is a left zero near- idempotent semigroup union of near – null semigroup.

Dually, we can prove that a right zero near- idempotent semigroup is a right zero - idempotent semigroup union of near null semigroup.

Lemma: Let S be a left zero near idempotent semigroup with a, b in S.

Then $\xi_{ab} = \xi_a$

Proof: Let $x, y, a, b \in S$

Then $xaby = xay$ for all $x, y \in S$

Hence $\xi_{ab} = \xi_a$

Theorem: Every left zero near – idempotent semigroup is a left zero union of near – null semigroups.

Theorem: Every right zero near – idempotent semigroup is a right zero union of near – null semigroup.

Example Let the set $S = \{ 1, 2\}$ where $1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ under the matrix multiplication modulo 2, we get the following multiplication table

	1	**2**
1	**1**	**1**
2	**1**	**1**

It displays the null semigroup.

Example: Let the set $S = \{ 1, 2, 3, 4\}$ under the operation $a * b = a^b \pmod 2$

	1	2	3	4
1	1	1	1	1
2	2	2	2	2
3	1	1	1	1
4	2	2	2	2

The above table displays the near null semigroup,
Here 1,2 are idempotents;3, 4 near idempotent elements
x.4.y = x.2
$x.4^2.y$ = x.2.y
= x.2

Conclusion: The concept of idempotency can be generalized in various ways. Near- idempotency is one such generalization. Jayalakshmi, A., and Ananth K. Atre[7] have generalized bands into semigroups which are left inflation of a band, a rectangular band, a right regular band etc; in each of these semigroups S^2 is a band. There is scope for other types of generalizations of a band which may throw new light on the structure of semigroups of this kind.

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