TOPOLOGIES GENERATED BY THE FUZZY SETS

R.PADMAPRIYA, P.THANGAVELU

Abstract: Zadeh [6] introduced the concept of fuzzy sets in 1965, as a generalisation of the classical sets. An important concept in fuzzy set theory is the notion of α-cuts that play a dominant role in studying the relationship between fuzzy sets and crisp sets. Given a fuzzy set A in X and any real number $\alpha \in [0, 1]$, the α-cut of A, denoted by ${}^{\alpha}A$ is the crisp set ${}^{\alpha}A = \{ x \in X : A(x) \geq \alpha \}$ and the strong α-cut denoted by ${}^{\alpha+}A$ is the crisp set given by ${}^{\alpha+}A = \{ x \in X : A(x) > \alpha \}$.The collection of α-cuts of a fuzzy set A of X may generate a topology on the underlying set X. Such a topology is called the topology generated by the α-cuts of A. Analogously the strong α-cut of A may generate a topology on X. In this paper the topology generated by the α-cuts and also topology generated by the strong α-cuts of a fuzzy set are studied.

Keywords: crisp set, fuzzy set, α-cuts, strong α-cuts.

Introduction: Zadeh [6] introduced the concept of fuzzy sets in 1965, as a generalisation of the classical sets. An important concept in fuzzy set theory is the notion of α-cuts that play a dominant role in studying the relationship between fuzzy sets and crisp sets. Given a fuzzy set A in X and any real number $\alpha \in [0, 1]$, the α-cut of A, denoted by ${}^{\alpha}A$ is the crisp set ${}^{\alpha}A = \{ x \in X : A(x) \geq \alpha \}$ and the strong α-cut denoted by ${}^{\alpha+}A$ is the crisp set given by ${}^{\alpha+}A = \{ x \in X : A(x) > \alpha \}$.The collection of α-cuts of a fuzzy set A of X may generate a topology on the underlying set X. Such a topology is called the topology generated by the α-cuts of A. Analogously the strong α-cut of A may generate a topology on X. In this paper the topology generated by the α-cuts and also topology generated by the strong α-cuts of a fuzzy set are studied.

Topology generated by the α-cuts:

Throughout this section τ_A denotes the topology generated by the α-cuts of a fuzzy subset A of X.

It is easy to check that the α-cuts of the

fuzzy set $A(n) = \begin{cases} 0, & n \text{ is even} \\ \dfrac{1}{(n+1)}, & n \text{ is odd} \end{cases}$ of

$X = \{1, 2, 3, \ldots\ldots.\}$

and the α-cuts of a fuzzy set

$A(n) = \begin{cases} 0, & n \text{ is even} \\ \dfrac{1}{2}, & n = 1 \\ \dfrac{1}{n}, & \text{otherwise} \end{cases}$ of

$X = \{ 1,2,3,\ldots.\}$generate topologies on X.

Remark : T(A) = The collection of all α-cuts of A.

Theorem 2.1: Every constant fuzzy subset of X generates the indiscrete topology on X.

Proof: Let $A: X \rightarrow [0, 1]$. If A(x) = 0 for all $x \in X$, then ${}^{0}A = X$ and ${}^{\alpha}A = \emptyset$ for $\alpha > 0$. If A(x) = 1 for all $x \in X$, then ${}^{0}A = X$ and ${}^{\alpha}A = X$ for $\alpha > 0$. Now fix $\alpha_0 \in (0,1)$. If $A(x) = \alpha_0$, for all $x \in X$, then ${}^{\alpha_0}A = X$. Suppose $\alpha > \alpha_0$,then ${}^{\alpha}A = \{ x : A(x) \geq \alpha \} = \{ x : \alpha_0 \geq \alpha \} = \emptyset$. Suppose $\alpha \leq \alpha_0$,

then ${}^{\alpha}A = \{x: A(x) \geq \alpha \} = \{ x : \alpha_0 \geq \alpha \} = X$.

Thus $\tau_A = \{ X , \emptyset\}$. This completes the proof.

Theorem 2.2 : If A is a fuzzy subset of X then $\tau_A = \{X , \emptyset\}$where $X = \{x\}$.

Proof: Suppose X = {x} ; Let $A = \dfrac{a}{x}$; $0 \leq a \leq 1$.

Then T (A) = { X }.

Topology generated by T (A) = Topology generated by {X} = {X , ∅}.

Therefore $\tau_A = \{X , \emptyset\}$.

Theorem 2.3: If A is a fuzzy subset of X = { x , y } then $\tau_A = \{ X , \emptyset\}$or $\tau_A = \{X , \emptyset, \lambda \}$ where $\lambda \in \{\{x\} ,\{y\}\}$.

Proof: Let X = { x , y }. There are 4 topologies on X. They are given by $\tau_1 = \{X , \emptyset\}$, $\tau_2 = \{X , \emptyset, \{x\} \}$, $\tau_3 = \{X , \emptyset, \{y\}\}$, $\tau_4 = \{X , \emptyset , \{x\} ,\{y\}\}$.

Case 1: If $A = \dfrac{a}{x} + \dfrac{a}{y}$ where $0 \leq a \leq 1$ then T (A) = {X}.

Topology generated by T(A)=Topology generated by {X} = {X , ∅}. $\tau_A = \{X, \emptyset\} = \tau_1$.

Case 2: If $A = \dfrac{a}{x} + \dfrac{b}{y}$ where $0 \leq b < a \leq 1$ then T (A) = { X ,{ x}}.

Topology generated by T (A)=Topology generated by { X , { x}} = { X , { x } , ∅}. $\tau_A = \{ X , \{ x \} , \emptyset\} = \tau_2$.

Case 3: If $A = \dfrac{a}{x} + \dfrac{b}{y}$ where $0 \leq a < b \leq 1$ then T (A) = {X ,{y }}.

Topology generated by T (A) = Topology generated by {X ,{y} = {X ,{y} , ∅}. $\tau_A = \{X , \{y\}, \emptyset\} = \tau_3$. This completes the proof.

Therefore we conclude that only 3 topologies on {x , y}are generated by the α-cuts of the fuzzy subsets of {x, y}.

Theorem 2.4: The discrete topology on {x ,y}is not generated by the α-cuts of any fuzzy subset of {x ,y}.

Proof : Let A be a fuzzy subset of {x , y}.

Then $A = \frac{a}{x} + \frac{b}{y}$ where $0 \le a , b \le 1$.

If a = b then by the previous Theorem ,

$\tau_A = \{X , \emptyset\}$ that is not discrete.

If a > b then by the previous Theorem,

$\tau_A = \{X , \{x\}, \emptyset \}$ that is not discrete.

If a < b then by the previous Theorem,

$\tau_A = \{X ,\{y\} , \emptyset\}$ that is not discrete. Thus we conclude that the discrete topology on { x , y } is not generated by the α-cuts of any fuzzy subset of { x , y }.

Theorem 2.5: If A is a fuzzy subset of X = {x , y , z } then $\tau_A = \{X , \emptyset\}$ or $\{X , \emptyset, \lambda_1\}$ or $\{X , \emptyset, \lambda_2\}$ or $\{X , \emptyset , \lambda_1, \lambda_2\}$ with $\lambda_1 \subseteq \lambda_2$ where $\lambda_1 \in \{\{x\},\{ y\},\{z\}\}$ and $\lambda_2 \in \{\{x , y\} ,\{y , z\} ,\{x , z\}\}$.

Proof : Analogous to Theorem 2.3.

Theorem 2.6: If A is a fuzzy subset of X = { x , y , z , w} then $\tau_A = \{X , \emptyset \}$ or $\{X , \emptyset , \lambda_1\}$ or $\{ X , \emptyset , \lambda_2\}$ or $\{X , \emptyset , \lambda_3\}$ or $\{X , \emptyset , \lambda_1 , \lambda_2\}$ with $\lambda_1 \subseteq \lambda_2$ or $\{X , \emptyset , \lambda_2 , \lambda_3 \}$ with $\lambda_2 \subseteq \lambda_3$ or $\{ X , \emptyset , \lambda_1 , \lambda_3\}$ with $\lambda_1 \subseteq \lambda_3$ or $\{X, \emptyset , \lambda_1 , \lambda_2 , \lambda_3\}$ with $\lambda_1 \subseteq \lambda_2 \subseteq \lambda_3$ where $\lambda_1 \in \{\{x\},\{y\}, \{ z \},\{w\}\}$, $\lambda_2 \in \{\{x , y\}, \{x , z\} ,\{x , w\}, \{y, z\},\{ y ,w\},\{z , w\}\}$ and $\lambda_3 \in \{\{x , y , z\},\{x , z , w\},\{x , y , w\}, \{y , z , w\}\}$.

Proof : Analogous to Theorem 2.3.

Remark: $T^+(A)$ = The collection of all strong α-cuts of A.

3. Topology generated by the strong α-cuts:

Throughout this section τ^+_A denotes the topology generated by the strong α-cuts of a fuzzy subset A of X.

It is easy to check that the strong α-cuts of a fuzzy set

$$A(x) = \begin{cases} \frac{3x-1}{8} & , 1 \le x \le 3 \\ 0 & , otherwise \end{cases}$$

of X = [1 , 3] generate a topology on X and the strong α-cuts of a fuzzy set A(x) = $\begin{cases} \frac{x-1}{2} & , 1 \le x \le 3 \\ 0 & , otherwise \end{cases}$ of X where X is the set of real numbers do not generate a topology on X however they generate a topology on the support of A.

Theorem 3.1: If A is a fuzzy subset of X then $\tau^+_A = \{X , \emptyset \}$ where X = {x}.

Proof: Suppose X = {x} ; Let $A = \frac{a}{x}$; $0 \le a \le 1$.Then $T^+ (A) = \{ X ,\emptyset \}$**.** Topology generated by $T^+(A)$ = Topology generated by $\{X, \emptyset \} = \{ X , \emptyset \}$. $\tau^+_A = \{ X ,\emptyset \}$.

Theorem 3.2: If A is a fuzzy subset of X ={x ,y} then $\tau^+_A = \{X , \emptyset \}$ or $\tau^+_A = \{X , \emptyset , \lambda \}$ where $\lambda \in \{\{x\},\{y\}\}$.

Proof: Let X = {x ,y}. There are 4 topologies on X. They are given by $\tau^+_1 = \{X , \emptyset \}$, $\tau^+_2 = \{ X , \emptyset, \{x\} \}$, $\tau^+_3 = \{ X , \emptyset ,\{y\} \}$, $\tau^+_4 = \{ X , \emptyset , \{x\},\{y\}\}$.

Case 1: If $A = \frac{a}{x} + \frac{a}{y}$ where $0 \le a \le 1$ then $T^+ (A) = \{ X, \emptyset \}$.Topology generated by $T^+ (A)$ = Topology generated by $\{X , \emptyset \}= \{X , \emptyset \}$**.** $\tau^+_A = \{X , \emptyset \} = \tau^+_{1.}$

Case 2: If $A = \frac{a}{x} + \frac{b}{y}$ where $0 \le b < a \le 1$ then $T^+ (A) = \{X, \{x\}, \emptyset \}$.

Topology generated by $T^+(A)$ = Topology generated by $\{X, \{x\}, \emptyset \} =\{X ,\{x\}, \emptyset \}$. $\tau^+_A =\{X ,\{x\},\emptyset \} = \tau^+_{2.}$

Case 3: If $A = \frac{a}{x} + \frac{b}{y}$ where $0 \le a < b \le 1$ then $T^+(A)=\{X,\{y\}, \emptyset \}$**.**Topology generated by $T^+(A)$= Topology generated by $\{X,\{y\},\emptyset \}= \{ X ,\{y\}, \emptyset \}$. $\tau^+_A =\{X , \{ y \},\emptyset \}= \tau^+_{3.}$ This completes the proof.

Therefore we conclude that Only 3 Topologies on X = {x, y} are generated by the strong α-cuts of the fuzzy subsets of {x ,y}.

Theorem 3.3: The discrete topology on {x, y} is not generated by the strong α-cuts of any fuzzy subset of {x, y}.

Proof : Let A be a fuzzy subset of {x , y}.

Then $A = \frac{a}{x} + \frac{b}{y}$ where $0 \le a , b \le 1$.

If a = b then by the previous Theorem, A generates the topology $\tau^+_A =\{X , \emptyset\}$ that is not discrete.

If a > b then by the previous Theorem, A generates the topology $\tau^+_A =\{X , \{x\}, \emptyset \}$ that is not discrete.

If a < b then by the previous Theorem, A generates the topology $\tau^+_A = \{X ,\{y\} , \emptyset\}$ that is not discrete.

Thus we conclude that the discrete topology on { x , y } is not generated by the strong α-cuts of any fuzzy subset of { x , y }.

Theorem 3.4: If A is a fuzzy subset of X = {x , y , z } then $\tau^+_A = \{X , \emptyset \}$ or $\{X , \emptyset, \lambda_1\}$ or $\{X, \emptyset , \lambda_2\}$ or $\{X, \emptyset , \lambda_1 , \lambda_2 \}$ with $\lambda_1 \subseteq \lambda_2$ where $\lambda_1 \in \{\{x\},\{y\}, \{ z\}\}$ and $\lambda_2 \in \{\{x , y \},\{y , z\} ,\{x ,z \}\}$.

Proof: Analogous to Theorem 3.2.

Theorem 3.5 : If A is a fuzzy subset of X = {x , y , z , w}then $\tau^+_A=\{ X ,\emptyset \}$ or $\{X , \emptyset, \lambda_1\}$ or $\{X , \emptyset , \lambda_2\}$ or $\{X, \emptyset, \lambda_3\}$ or $\{X , \emptyset, \lambda_1, \lambda_2\}$ with $\lambda_1 \subseteq \lambda_2$ or {X ,

$\emptyset, \lambda_2, \lambda_3\}$ with $\lambda_2 \subseteq \lambda_3$ or$\{X, \emptyset, \lambda_1, \lambda_3\}$with $\lambda_1 \subseteq \lambda_3$ or $\{X, \emptyset, \lambda_1, \lambda_2, \lambda_3\}$ with $\lambda_1 \subseteq \lambda_2 \subseteq \lambda_3$ where $\lambda_1 \in \{\{x\}, \{y\}, \{z\}, \{w\}\}$, $\lambda_2 \in \{\{x, y\}, \{x, z\}, \{x, w\}, \{y, z\}, \{y, w\}, \{z, w\}\}$ and $\lambda_3 \in \{\{x, y, z\}, \{x, z, w\}, \{x, y, w\}, \{y, z, w\}\}$.

Proof : Analogous to Theorem 3.2.

4. Comparison of τ_A with τ^+_A

Theorem 4.1 : τ_A is finer than τ^+_A if the strong α-cuts of A form a base for a topology on X.

Proof: Let τ_A denote the topology generated by the α-cuts of fuzzy subset A of X. Let τ^+_A denote the topology generated by the strong α-cuts of fuzzy subset A of X. To prove τ_A is finer than τ^+_A.

For, let $x \in X$ and $\beta \in [0, 1]$ with $x \in {}^{\beta+}A$. Therefore $A(x) > \beta$. Take $\beta_1 = A(x)$. Then $x \in {}^{\beta_1}A$. Claim ${}^{\beta_1}A \subseteq {}^{\beta+}A$. Let $y \in {}^{\beta_1}A \Rightarrow A(y) \geq \beta_1 \Rightarrow A(y) \geq A(x) \Rightarrow A(y) \geq \beta \Rightarrow y \in {}^{\beta+}A$. Hence by lemma we proved τ_A is finer than τ^+_A.

Theorem 4.2 : τ^+_A is not finer than τ_A.

Proof : We will prove this by an example. Let X be the set of real numbers.

$$A(x) = \frac{x-1}{2}, \quad 1 \leq x \leq 3$$

For $0 \leq \alpha \leq 1$, Let $A(x) \geq \alpha \Rightarrow \frac{x-1}{2} \geq \alpha \Rightarrow x \geq 2\alpha + 1 \Rightarrow 2\alpha + 1 \leq x \Rightarrow x \in [2\alpha + 1, 3]$.

Therefore ${}^{\alpha}A = [2\alpha+1, 3]$ and ${}^{\alpha+}A = (2\alpha + 1, 3]$. So it is obvious that τ^+_A is not finer than τ_A.

Theorem 4.3 : Topology generated by the α-cuts of a nonzero fuzzy subset A of X is same as the Topology generated by the strong α-cuts of A of X if and only if X is a finite set .

Theorem 4.4 : Out of all topologies on X = { x , y , z , w} generated by the fuzzy subsets of X ,

$\tau_A = \{X, \emptyset, \lambda_1, \lambda_2, \lambda_3\}$ with $\lambda_1 \subseteq \lambda_2 \subseteq \lambda_3$ where $\lambda_1 \in \{\{x\}, \{y\}, \{z\}, \{w\}\}$, $\lambda_2 \in \{\{x, y\}, \{x, z\}, \{x, w\}, \{y, z\}, \{y, w\}, \{z, w\}\}$ and $\lambda_3 \in \{\{x, y, z\}, \{x, z, w\}, \{x, y, w\}, \{y, z, w\}\}$ is T_0 and all other topologies generated by the fuzzy subsets of X,are not T_0.

Theorem 4.5: None of the topology generated by the fuzzy subset of X with $|X| = 4$ is T_1 .

Proof : From Theorem 4.4, all the topologies given in the list except $\tau_A = \{X, \emptyset, \lambda_1, \lambda_2, \lambda_3\}$with $\lambda_1 \subseteq \lambda_2 \subseteq \lambda_3$ are not T_0 and hence are not T_1 . So it is enough to verify the topology $\tau_A = \{X, \emptyset, \lambda_1, \lambda_2, \lambda_3\}$ with $\lambda_1 \subseteq \lambda_2 \subseteq \lambda_3$. Suppose $\lambda_1 = \{x\}$, $\lambda_2 = \{x, y\}$ and $\lambda_3 = \{x, y, z\}$ then $\tau_A = \{X, \emptyset, \{x\}, \{x, y\}, \{x, y, z\}\}$. If we consider any pair, we cannot find an open set which satisfies T_1 .This completes the proof.

Theorem 4.6 : Topology generated by the fuzzy subset A of X = { x , y , z } is T_0 if and only if the membership grades of A are all distinct.

Theorem 4.7: The α-cut of a complement of a fuzzy subset A of X is closed in (X, τ^+_A).

Proof: Let A be any fuzzy subset of X such that τ_A and τ^+_A are the topologies generated by the α-cuts and strong α-cuts of the fuzzy subset of A. Using theorem 2.1 (v) from George J. Klir/Bo Yuan , ${}^{\alpha}A'$= The complement of ${}^{(1-\alpha)+}A$ in $X = X - {}^{(1-\alpha)+}A$. Since ${}^{(1-\alpha)+}A$ is open in (X, τ^+_A) , ${}^{\alpha}A'$ is closed in (X, τ^+_A).

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R.Padmapriya, Assistant Professor/ Department of Mathematics /
Nandha College of Technology, Erode 638052- India / priyabharathi28@gmail.com
P.Thangavelu, Professor/ Department of Mathematics / Karunya University,
Coimbatore 641114 - India / ptvelu12@gmail.com