ON A CERTAIN CLASS OF UNIFORMLY CONVEX UNIVALENT FUNCTIONS USING RUSCHEWEYGH DIFFERENTIAL OPERATOR

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Abstract: In this paper, using Ruscheweygh differential operator, we have introduce new class of univalent uniformly convex functions in the unit disc D: = {z ∈ C: |z| < 1} and obtain the coefficient bounds, extreme bounds and radius of starlikeness for the functions belonging to this generalized class. The various results obtained in this paper are sharp.

Keywords: Analytic and univalent functions, Ruscheweygh derivative, λ convex functions, Coefficients bounds, etc.

Introduction: Let S be the class of analytic functions defined on the open unit disk D: = {z ∈ C: |z| < 1} and normalized by the conditions f (0) = 0 and f ′ (0) = 1. A function $f \in S$ has Taylor's series expansion of the form

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ (1.1)

G.Murugusundaramoorthy[8], Goodman [9],[10] and Ronning [11], [12] have studied the following subclasses.

1. A function f(z) ∈ S is said to be in the class S $(\propto, \beta)$ of uniformly β starlike function if it satisfies the condition $\mathrm{Re}\left\{1 + \frac{f''(z)}{f'(z)} - \propto\right\} > \beta \left|\frac{zf'(z)}{f(z)} - 1\right|$ (1.2) where $-1 < \propto \leq 1$ *and* $\beta \geq 0$.
2. A function f(z) ∈ *S* is said to be in the class *UCV* $(\propto, \beta)$ of uniformly convex function if it satisfies the condition $\mathrm{Re}\left\{1 + \frac{f''(z)}{f'(z)} - \propto\right\} > \beta \left|\frac{zf''(z)}{f'(z)} - 1\right|$ (1.3)

It follows from (1.2) and (1.3) that

f(z) ∈ $UCV(\propto, \beta)$ is equivalent to $zf'(z) \in S(\propto, \beta)$

Given two functions f, g ∈ S, where f (z) is given by (1.1) and g(z) is given by

$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ (1.4)

The Hadamard product of f and g, denoted by f*g and is defined by

$f*g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$ (1.5)

The Ruscheweygh derivative of order k is denoted by $D^k f$ and is defined as follows: If

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then

$D^k f(z) = \frac{z}{(1-z)^{k+1}} * f(z)$,where $*$ denotes the Hadamard product of two analytic functions

We have,

$$D^k f(z) = \frac{z}{(1-z)^{k+1}} * f(z) = z + \sum_{n=2}^{\infty} B_n(k) a_n z^n , \; k > -1, \; z \in D,$$

where,

$$B_n(k) = \frac{(k+1)(k+2)\ldots(k+n-1)}{(n-1)!}$$

Let

$$D^*(\propto, \beta)) = \left\{ f(z) \in S : \left| \frac{\frac{z(D^k f(z))'}{D^k f(z)} - 1}{\frac{\beta z(D^k f(z))'}{D^k f(z)} - \propto} \right| < \mu \right\} \quad (1.6)$$

where, $-1 \leq \beta \leq \propto \leq 1$,$0 < \mu \leq 1$ and $D^k f$ represents Ruscheweygh derivative of order k.

Let T denote the subclass of A consisting of functions of the form

$f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ $(a_n \geq 0)$. (1.7)

Let $D^{**}(\propto, \beta) = D^*(\propto, \beta) \cap T$.

Properties of the class $D^*(\propto, \beta)$:

Theorem 2.1: A function f defined by (1.1) is in the class $D^*(\propto, \beta)$ if

$\sum_{n=2}^{\infty} B_n(k)[(n-1) + \mu(n\beta - \propto)]|a_n| < \mu(\beta - \propto)$ (2.1)

Proof: Let $|z| = 1$,

$$\text{Since} \left| \frac{\frac{z(D^k f(z))'}{D^k f(z)} - 1}{\frac{\beta z(D^k f(z))'}{D^k f(z)} - \propto} \right| = \left| \frac{\frac{z + \sum_{n=2}^{\infty} nB_n(k) a_n z^n}{z + \sum_{n=2}^{\infty} B_n(k) a_n z^n} - 1}{\frac{\beta(z + \sum_{n=2}^{\infty} nB_n(k) a_n z^n)}{z + \sum_{n=2}^{\infty} B_n(k) a_n z^n} - \propto} \right|$$

$$= \left| \frac{\sum_{n=2}^{\infty}(n-1)B_n(k) a_n z^n}{(\beta - \propto)z + \sum_{n=2}^{\infty}(n\beta - \propto)B_n(k) a_n z^n} \right|$$

$$\leq \frac{\sum_{n=2}^{\infty}(n-1)B_n(k)|a_n||z^n|}{(\beta - \propto)|z| - \sum_{n=2}^{\infty}(n\beta - \propto))B_n(k)|a_n||z^n|}$$

$$\leq \frac{\sum_{n=2}^{\infty}(n-1)B_n(k)|a_n|}{(\beta - \propto) - \sum_{n=2}^{\infty}(n\beta - \propto))B_n(k)|a_n|} \quad (2.2)$$

Therefore, if f (z) satisfies the inequality (2.1), then the inequality (2.2) is less than μ and hence by maximum modulus theorem f(z) ∈ $D^*(\propto, \beta)$.

Theorem 2.2: A function f is in the class $D^{**}(\propto, \beta)$ if and only if

$\sum_{n=2}^{\infty} B_n(k)[(n-1) + \mu(n\beta - \propto)]|a_n| \leq \mu(\beta - \propto)$ (2.1)

Proof: In view of theorem (2.1), we need only to prove the necessary part.

If f(z) $D^{**}(\propto, \beta)$then,

$$\left| \frac{\frac{z(D^k f(z))'}{D^k f(z)} - 1}{\frac{\beta z(D^k f(z))'}{D^k f(z)} - \propto} \right| < \mu$$

$$\text{i.e} \left| \frac{\sum_{n=2}^{\infty}(n-1)B_n(k) a_n z^n}{(\beta - \propto)z + \sum_{n=2}^{\infty}(n\beta - \propto)B_n(k) a_n z^n} \right| < \mu$$

Since $|\mathrm{Re}\{z\}| \leq |z|$, we have

$$\mathrm{Re}\left\{ \frac{\sum_{n=2}^{\infty} nB_n(k) a_n z^{n-1}}{(\beta - \propto)z - \sum_{n=2}^{\infty}(n\beta - \propto)B_n(k) a_n z^n} \right\} < \mu$$

The above condition must hold for all values of z; |z| = r < 1.

Upon choosing the value of z to be real and letting $z \to 1^-$, we get

$\sum_{n=2}^{\infty} B_n(k)[(n-1) + \mu(n\beta - \propto)]|a_n| \leq \mu(\beta - \propto)$

Corollary 2.1: If f (z) ∈ $D^*(\propto, \beta)$

$$|a_n| \leq \frac{\mu(\beta - \propto)}{B_n(k)[(n-1) + \mu(n\beta - \propto)]} \quad (n = 2, 3, \ldots).$$

Proof: By theorem (2.1) , if $f(z) \in D^{*}(\propto,\beta)$ then

$\sum_{n=2}^{\infty} B_n(k)[(n-1)+\mu(n\beta-\propto)]|a_n| \leq \mu(\beta-\propto)$.

$\Rightarrow |a_n| \leq \frac{\mu(\beta-\propto)}{B_n(k)[(n-1)+\mu(n\beta-\propto)]}$ (n=2,3,…).

Theorem: 2.3 Let f(z) defined by (1.1) and g(z) defined by (1.2) be in the class $D^{**}(\propto,\beta)$. Then the function $h(z) = \xi f(z) + (1-\xi)g(z) = z + \sum_{n=2}^{\infty} c_n z^n$,where $c_n = \xi a_n + (1-\xi) b_n$, ($0 \leq \xi \leq 1$) belongs to the class $D^{**}(\propto,\beta)$.

Proof: Since f(z) and g(z) $\in D^{**}(\propto,\beta)$, we have

$\sum_{n=2}^{\infty} B_n(k)[(n-1)+\mu(n\beta-\propto)]|a_n| \leq \mu(\beta-\propto)$

and

$\sum_{n=2}^{\infty} B_n(k)[(n-1)+\mu(n\beta-\propto)]|b_n| \leq \mu(\beta-\propto)$

Clearly, $h(z) = \xi f(z) + (1-\xi)g(z)$

$= z + \sum_{n=2}^{\infty} c_n z^n$,where

$c_n = \xi a_n + (1-\xi) b_n$.

Consider

$\sum_{n=2}^{\infty} B_n(k)[(n-1)+\mu(n\beta-\propto)]|c_n|$

$= \sum_{n=2}^{\infty} B_n(k)[(n-1)+\mu(n\beta-\propto)]|\xi a_n + (1-\xi)b_n|$

$\leq \sum_{n=2}^{\infty} B_n(k)[(n-1)+\mu(n\beta-\propto)]|a_n||\xi| + \sum_{n=2}^{\infty} B_n(k)[(n-1)+\mu(n\beta-\propto)]|b_n||(1-\xi)|$

$\leq \mu(\beta-\propto)\xi + \mu(\beta-\propto)(1-\xi) = \mu(\beta-\propto)$

Therefore $h(z) \in D^{**}(\propto,\beta)$

Thus the set $D^{**}(\propto,\beta)$ is closed under convex linear combination.

Theorem 2.4(Extreme Points).

Let $f_1(z) = z$ and for n =2,3, 4, …

$f_n(z) = z - \frac{\mu(\beta-\propto)}{B_n(k)[(n-1)+\mu(n\beta-\propto)]} z^n$.

Then $f(z) \in D^{**}(\propto,\beta)$ if and only if f(z) can be expressed as $f(z) = \sum_{n=1}^{\infty} \xi_n f_n(z)$ where $\xi_n \geq 0$ and $\sum_{n=1}^{\infty} \xi_n = 1$.

Proof: Suppose that $f(z) = \sum_{n=1}^{\infty} \xi_n f_n(z)$, then

$f(z) = z - \sum_{n=2}^{\infty} \frac{\mu(\beta-\propto)}{B_n(k)[(n-1)+\mu(n\beta-\propto)]} \xi_n z^n$

$= z - \sum_{n=2}^{\infty} c_n z^n$, where,

$$c_n = \frac{\mu(\beta-\propto)}{B_n(k)[(n-1)+\mu(n\beta-\propto)]} \xi_n$$

Consider

$\sum_{n=2}^{\infty} B_n(k)[(n-1)+\mu(n\beta-\propto)]c_n$

$= \sum_{n=1}^{\infty} B_n(k)[(n-1)+\mu(n\beta-\propto)] * \left[\frac{\mu(\beta-\propto)}{B_n(k)[(n-1)+\mu(n\beta-\propto)]}\right]$.

$= \mu(\beta-\propto)\sum_{n=2}^{\infty} \xi_n \leq \mu(\beta-\propto)$,

Since $0 \leq \sum_{n=2}^{\infty} \xi_n \leq 1$.

Hence $f(z) \in D^{**}(\propto,\beta)$.

Conversely, suppose that

$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in D^{**}(\propto,\beta)$.

Therefore we have by corollary (2.1)

$a_n \leq \frac{\mu(\beta-\propto)}{B_n(k)[(n-1)+\mu(n\beta-\propto)]}$

Setting $\xi_n = \frac{B_n(k)[(n-1)+\mu(n\beta-\propto)]}{\mu(\beta-\propto)} a_n$ for n= 2,3,4,5,… and $\xi_1 = 1 - \sum_{n=2}^{\infty} \xi_n$

We see that $\xi_n \geq 0$, for $n = 2,3,\ldots$ and

$\sum_{n=2}^{\infty} \xi_n = \sum_{n=2}^{\infty} \frac{B_n(k)[(n-1)+\mu(n\beta-\propto)]}{\mu(\beta-\propto)} a_n \leq 1$,

since $f(z) \in D^{**}(\propto,\beta)$.

$\xi_1 = 1 - \sum_{n=2}^{\infty} \xi_n \geq 0$.Thus $\xi_n \geq 0$ for n = 1, 2, 3,… and $\sum_{n=1}^{\infty} \xi_n = 1$

Now $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$

$= z - \sum_{n=2}^{\infty} \frac{\mu(\beta-\propto)}{B_n(k)[(n-1)+\mu(n\beta-\propto)]} z^n \xi_n$

$= \sum_{n=1}^{\infty} \xi_n f_n(z)$.

Hence the proof.

Closure Theorem:

Theorem 3.1: Let the function f_i defined by

$f_i(z) = z - \sum_{n=2}^{\infty} a_{n,i} z^n (a_{n,i} \geq 0)$ be in the class $D^{**}(\propto_i,\beta_i)$ where $-1 \leq \beta_i \leq \propto_i \leq 1$ for each i = 1,2,…,m.

Then the function h defined by

$h(z) = z - \frac{1}{m}\sum_{n=2}^{\infty}(\sum_{i=1}^{m} a_{n,i}) z^n$ is in the class $D^{**}(\propto,\beta)$where $\propto = \min_{1\leq i\leq m}\{\propto_i\}$and $\beta = \max_{1\leq i\leq m}\{\beta_i\}$ (2.3)

Proof: Since $f_i(z) \in D^{**}(\propto_i,\beta_i)$

$$\sum_{n=2}^{\infty} B_n(k)[(n-1)+\mu(n\beta_i-\propto_i)]|a_{n,i}| \leq \mu(\beta_i-\propto_i).$$

Consider

$\sum_{n=2}^{\infty} B_n(k)[(n-1)+\mu(n\beta_i-\propto_i)]\left|\frac{1}{m}\sum_{i=1}^{m} a_{n,i}\right|$

$= \frac{1}{m}\sum_{i=1}^{m}\sum_{n=1}^{\infty}(\sum_{n=2}^{\infty} B_n(k)[(n-1)+\mu(n\beta_i-\propto_i)])|a_{n,i}|$

$$\leq \frac{1}{m}\sum_{i=1}^{m}\mu(\beta_i-\propto_i)$$

$\leq \frac{1}{m}\sum_{i=1}^{m}\mu(\beta-\propto) = \mu(\beta-\propto)$.

Hence by Theorem 2.2, we have

$h(z) \in D^{**}(\propto,\beta)$, where α and β is given by (2.3).

This completes the proof of the theorem.

Next we prove the theorem for the radius of starlikeness and convexity.

Radius of Starlikeness and Convexity:

Theorem 4.1

Let $f(z) \in D^{**}(\propto,\beta)$ Then

f(z) is starlike of order $\delta (0 \leq \delta < 1)$ in the disc $|z| \leq r$, where

$$r = \inf_{n\geq 2}\left\{\frac{(n-1)a_n z^n + \mu(n\beta-\propto)](2-1-\delta)B_n(k)}{\mu(\beta-\propto)(2-n-\delta)}\right\}^{\frac{1}{n-1}}$$

The results are sharp for the function

$f(z) = z - \frac{\mu(\beta-\propto)}{B_n(k)[(n-1)+\mu(n\beta-\propto)]} z^n$

(n = 2, 3, ⋯)

Proof: If a function f(z) is starlike of order $\delta(0 \le \delta < 1)$, then we have $\text{Re}\left\{\frac{z f'(z)}{f(z)}\right\} > \delta$.

That is $\left|\frac{z f'(z)}{f(z)} - 1\right| < 1 - \delta$.

Now $\left|\frac{z f'(z)}{f(z)} - 1\right| \le \left|\frac{\sum_{n=2}^{\infty}(n-1)a_n z^n}{z - \sum_{n=2}^{\infty} a_n z^n}\right| < 1 - \delta$

$\Rightarrow |\sum_{n=2}^{\infty}(n-1)a_n z^n| < (1 -.\delta)\times$
$|z - \sum_{n=2}^{\infty} a_n z^n|$.

Hence $\sum_{n=2}^{\infty} \frac{(2-n-\delta)}{1-\delta} a_n |z|^{n-1} < 1$

We note that $f(z) \in D^{**}(\propto, \beta)$ if and only if

$\sum_{n=2}^{\infty} \frac{B_n(k)[(n-1)+\mu(n\beta-\propto)]|a_n|}{\mu(\beta-\propto)} < 1$

Hence we get

$\frac{(2-n-\delta)}{1-\delta}|z|^{n-1} < \frac{B_n(k)[(n-1)+\mu(n\beta-\propto)]}{\mu(\beta-\propto)}$

Thus $|z| \le r = \inf_{n\ge2}\left\{\frac{(n-1)a_n z^n + \mu(n\beta-\propto)](2-1-\delta)B_n(k)}{\mu(\beta-\propto)(2-n-\delta)}\right\}^{\frac{1}{n-1}}$

for n= 2,3,4,… which proves starlikeness of family.

Theorem 4.2

Let $f(z) \in D^{**}(\propto, \beta)$

Then f(z) is convex of order $(0 \le \delta < 1)$ in the disc

$|z| \le r$,where

$r = \inf_{n\ge2}\left\{\frac{(n-1)a_n z^n + \mu(n\beta-\propto)](2-1-\delta)B_n(k)}{n\mu(\beta-\propto)(2-n-\delta)}\right\}^{\frac{1}{n-1}}$

The results are sharp for the function

$f(z) = z - \frac{\mu(\beta-\propto)}{B_n(k)[(n-1)+\mu(n\beta-\propto)]} z^n$

(n = 2,3, ⋯)

Proof: A function f(z) is convex of order δ $(0 \le \delta < 1)$ if and only if $\text{Re}\left\{1 + \frac{z f''(z)}{f'(z)}\right\} > \delta$ that is $\left|\frac{z f''(z)}{f'(z)}\right| < 1 - \delta$

Thus $\left|\frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{z - \sum_{n=2}^{\infty} a_n\, n\, z^{n-1}}\right| < 1\text{-}\,\delta$

That is $\sum_{n=2}^{\infty} \frac{n(2-n-\delta)}{1-\delta} a_n |z|^{n-1} < 1$

We note that $f(z) \in D^{**}(\propto, \beta)$ if and only if

$\sum_{n=2}^{\infty} \frac{B_n(k)[(n-1)+\mu(n\beta-\propto)]|a_n|}{\mu(\beta-\propto)} < 1$

Hence we get

$\frac{n(2-n-\delta)}{1-\delta}|z|^{n-1} < \frac{B_n(k)[(n-1)+\mu(n\beta-\propto)]}{\mu(\beta-\propto)}$

Thus

$|z| \le r = \inf_{n\ge2}\left\{\frac{(n-1)a_n z^n + \mu(n\beta-\propto)](2-1-\delta)B_n(k)}{n\mu(\beta-\propto)(2-n-\delta)}\right\}^{\frac{1}{n-1}}$ for

n = 2, 3, 4, … which proves the convex property of the family.

Conclusion: This paper gives the some geometric properties of some subclass of univalent functions.

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