# A STUDY ON PARAMETER FREE NUMERICAL METHOD FOR SINGULARLY PERTURBED LINEAR CONVECTION DIFFUSION PROBLEMS

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**Abstract:** In this paper, the authors discuss various standard techniques for the numerical solution of singular perturbed Linear Convection Diffusion problem in our dimension with Dirichlet boundary conditions of the form

$L_\varepsilon U_\varepsilon : \varepsilon u'' + a(x)u'_\varepsilon = f(x); x \epsilon \Omega = (0,1); u_\varepsilon(0) = A, u_\varepsilon(1) = B$

$a, f \in C^2(\Omega), a(x) \geq \alpha > 0, x \in [0,1]$

and we illustrate the importance of Layer resolving parameter free method on non uniform mesh.

**Keywords:** Boundary layer, Non –uniform mesh, Singular perturbation, Upwind finite difference method.

**Introduction:** The study of perturbation problem is important as they arises in several branches of engineering and applied mathematics. The word "perturbation" means a small distribution in a physical system. Mathematically, "perturbation method" is a method for obtaining approximate solution to complex equations (algebraic or differential) for which exact solution is not easy to find. Mainly,such problems which contain at least one small parameter $\varepsilon$ known as the perturbation parameter. We generally denote $\varepsilon$ for the effect of small disturbance in physical system and $\varepsilon$ is significantly less than unity.

Consider a differential equation $f\left(x, y, \frac{dy}{dx}, \epsilon\right) = 0$ with initial or boundary condition where $x$ is independent variable, $y$ is dependent variable and $\varepsilon \ll 1$ is the small parameter.

Depending upon the nature of effects, the perturbations can be divided into two

- Regular Perturbation
- Singular Perturbation

The aim of perturbation theory is to determine the behavior of the solution $S_\varepsilon$ of $P_\varepsilon$ as $\rightarrow 0$ . i.e to determine whether $S_\varepsilon$ converges to the solution of the problem as $\varepsilon \rightarrow 0$. Because the exact solution of a non trivial problem involving a singularly perturbed differential equation is usually unknown we need to compute numerical approximations to it. It may not be too difficult to construct a numerical method for a given single value of the singular perturbation parameter $\varepsilon$ say $\varepsilon = 10^{-6}$ , but the resulting numerical method may not be suitable for other values of the parameter say $\varepsilon = 10^{-4}$ or $\varepsilon = 10^{-10}$

Thus we are interested in constructing numerical methods that generate numerical solution which converge for all values of the parameter $\varepsilon$. Such numerical methods are called parameter-uniform or $\varepsilon$- uniform methods. Moreover the numerical method must be layer-resolving i,e. it should be able to detect the presence of a boundary layer.

**Motivation**: The development of small parameter methods led to the efficient use of boundary layer theory in various field of applied mathematics, for instance, fluid mechanics, fluid dynamics, elasticity, quantum mechanics, plasticity, chemical-reactor theory, aerodynamics, plasma dynamics, magneto hydrodynamics, rarefied-gas dynamics, oceanography, meteorology, diffraction theory, reaction diffusion processes, non equilibrium and radiating flows and other domains of the great world of fluid motion. In this section we give some singular perturbation models which arise in some of the above –mentioned areas, also some models described in the book by Mortan [1] which arise in quite distinct engineering and scientific fields.

(1)Consider the p-n junction with piecewise constant doping. With an appropriate scaling, the resulting one-Dimensional stationary [2] problem is given by

$$\varepsilon e^{\cdot} = n - p - 1$$
$$\varepsilon \dot{p} = -pe - \frac{\varepsilon}{2} j, p(1) = \frac{\delta^2}{n(1)}$$
$$\varepsilon n^{\cdot} = ne + \frac{\varepsilon}{2} j, n(0) = p(0), n(1) = p(1) + 1$$
$$\varepsilon \psi^{\cdot} = e, \psi(0) = 0$$

on $0 \leq x \leq 1$, where the small positive parameter $\varepsilon$ represents a scaled Debye length (typically bout $10^{-7/2}$ ) and where J and $\delta$ are fixed positive constants representing the current density and an intrinsic carrier concentration .

Also, $e$ is the electric field and p and n denote hole density and electron density , respectively, and both are positive.

(2) Consider the one –dimensional Schrödinger equation [3].

$$\varepsilon^2 \frac{d^2 \psi_\varepsilon}{dx^2} + \left(\lambda_\varepsilon - V(x)\right)\psi_\varepsilon = 0, \left||\psi_\varepsilon|\right| = 1$$

with $V(x)$ (potential) continuous and leading to $+\infty$ as $|x| \rightarrow \infty$ and with $\varepsilon = h\frac{(2m)^{1/2}}{2\pi}$,$h$ denoting the Planck's constant and m the mass. Associated to these ODEs, the eigen value problem in the Hilbert space $L^2$ $(-\infty, +\infty)$ with the norm $\left||u|\right| = [\int_{-\infty}^{+\infty} u^2(x)dx]^{\frac{1}{2}}$ has a discrete spectrum with Eigen values

$$\lambda_{\varepsilon}, 1, \lambda_{\varepsilon}, 2, \dots \dots . \dots . . \lambda_{\varepsilon}, n, \dots \dots ..$$

And Eigen functions $\psi_{\varepsilon}, 1\ \psi_{\varepsilon}, 2 \dots \dots . . \psi_{\varepsilon}, n \dots \dots$

3) The following equation [4] represents a time independent Fokker Planck equation for a one – dimensional dynamical system with state-independent random perturbations:

$$\varepsilon^2 \frac{d^2 \emptyset}{dx^2} + b(x)\frac{d\emptyset}{dx} = 0,$$
$$0 < \varepsilon \ll 1, x \in (0,1),$$

$\emptyset(1,\varepsilon) = A,\ \emptyset(1,\varepsilon) = B$

interval [0,1] and that $b(\gamma) = 0$ for some $0 < \gamma < 1$, the above problem is a resonant turning point problem.

**Preliminaries**: We introduce the following notation for finite difference operators. Let $P_\varepsilon$ be a one dimensional singularly perturbed differential equation defined on **Ω.** Let $u_\varepsilon$ be the exact solution of $P_\varepsilon$, then on an arbitrary mesh on $\Omega^N = \{x_i\}_{i=1}^N$ , we have

$$D^+u\,(x_i) = \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i}$$

$$D^-u\,(x_i) = \frac{u(x_i) - u(x_{i-1})}{x_i - x_{i-1}}$$

$$D^0\,u(x_i) = \frac{u(x_{i+1}) - u(x_{i-1})}{x_{i+1} - x_{i-1}}$$

Where $D^+$, $D^-$ , and $D^0$, are the forward, backward and central finite difference operators. The second order central difference operator $\delta^2$ is defined by

$$\delta^2 u(x_i) = \frac{2\left(D^+u(x_i) - D^-u(x_i)\right)}{x_{i+1} - x_{i-1}}$$

Let $u_\varepsilon$ be the exact solution of $P_\varepsilon$ and $(U_\varepsilon)\Omega^N$ be the approximate sequence of numerical solutions obtained through finite difference method where

$\overline{\Omega^N} = \{x_i\}_{i=1}^N$

Is the arbitrary mesh on which $U_\varepsilon$ is defined. Then the error $E_\varepsilon^N$ is defined by

$E_\varepsilon^N = \left|\left|U_\varepsilon - u_\varepsilon\right|\right|\overline{\Omega^N}$

And the corresponding $\varepsilon-$ uniform error $E^N$ is defined by $\quad E^N = max_\varepsilon E_\varepsilon^N$

For singular perturbation problems error is measured in the maximum norm.

**Illustrations**:

- Problem A

$$\varepsilon u'' + 2u' = 0$$
$$u(0) = 1, u(1) = 0$$

Its exact solution is given by

Where $b(x)$ denotes a gradient field. Under the assumptions that $b'(x)$ is strictly negative throughout the

$$u(x) = \frac{e^{\left(\frac{-2x}{\varepsilon}\right)} - e^{\left(\frac{-2}{\varepsilon}\right)}}{1 - e^{\left(\frac{-2}{\varepsilon}\right)}}$$

- Problem B

$-\varepsilon u^{\wedge\prime\prime}\,(x) + (x-1)\,u^{\wedge\prime}\,(x) = f\ u(0) = 0\,, u(1) = 0$

Where $f$ is chosen such that

$u(x) = \frac{1-e^{-x/\varepsilon}}{1-e^{-1/\varepsilon}} - sin\frac{\pi x}{2}$ be the exact solution.

**Centered finite difference method:** In this section we consider the classical centered finite difference method on a uniform mesh for by discretizing the differential operator

$L_\varepsilon\ \varepsilon\delta^2 u_\varepsilon(x_i) + a(x_i)D^0 u_\varepsilon(x_i) = f(x_i);\ x_i \in \Omega^N$

Where $\overline{\Omega}^N$ is the uniform mesh

$\overline{\Omega^N} = \{x_i / x_i = i/N,\ i = 0, \ldots\ldots, N\}$

Applying Centered finite difference method to Problem A, we get

$(\varepsilon + h)u_{i+1} - 2\varepsilon u_i + (\varepsilon - h)u_{i-1} = 0\ \ u_0 = 1, u_n = 0$

Similarly from problem B,

$$(b(i)h - 2\varepsilon)u_{i+1} + (4\varepsilon)u_i + (-2\varepsilon - b(i)h)u_{i-1} = 2h^2 f(i), u_0 = 0, u_n = 0$$

In the following Fig.1(a)&(b), Fig.2(a)&(b) we compare the exact solutions and the resulting numerical solutions for N = 16 and ε =0.01 and 0.00001

We see that the numerical solutions have large non-physical oscillations, which increases unboundedly as $\varepsilon$ decreases. In the first figure the numerical solutions oscillates between positive and negative values. This is in contrast to the behavior of the exact solution which decreases monotonically from 1 to 0.We conclude from these numerical experiments that Centered finite difference method on uniform mesh is not satisfactory for the Problem.

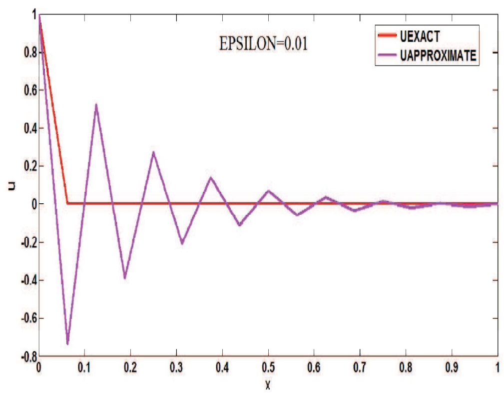


Fig.1(a)

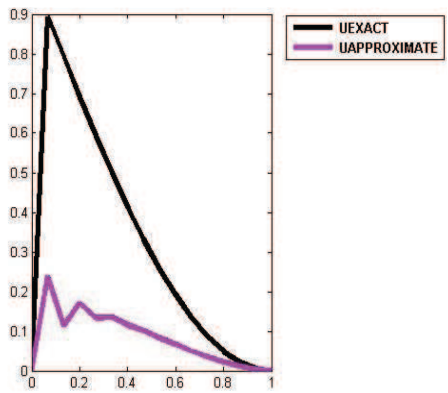


Fig.1(b)

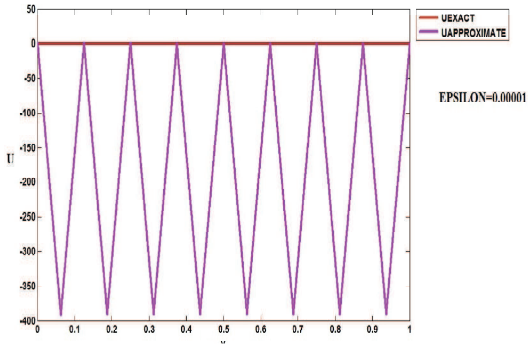


Fig.2(a)

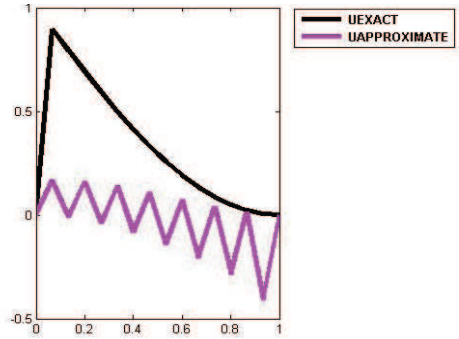


Fig.2(b)

**Upwind finite difference method**: Here we consider an upwind difference method on a uniform mesh where the first derivative term is approximated by a discrete derivative; which uses mesh points only in the upwind direction from that mesh point.$\varepsilon\delta^2 u_\varepsilon(x_i) + a(x_i)D^+u_\varepsilon(x_i) = f(x_i);\ x_i\epsilon\Omega^N$ where $\overline{\Omega^N}$ is the uniform mesh

$$\overline{\Omega^N} = \{x_i/x_i = i/N,\ i = 0,\ldots\ldots,N\}$$

Consider the convection-diffusion problem A, Applying Upwind finite difference method, the difference approximation to the differential equation is given by

$$(\varepsilon + 2h)u_{i+1} - (2\varepsilon + 2h)u_i + \varepsilon u_{i-1} = 0,$$

$$u_0 = 1, u_n = 0$$

Similarly, problem B, becomes

$$(-\varepsilon)u_{i+1} + (2\varepsilon + b(i)h)u_i + (-b(i)h - \varepsilon)u_{i-1} = f(i)h^2, u_0 = 0, u_n = 0$$

Solving this system of algebraic equations we get the numerical solution to the differential equation. The following Fig.3(a)&(b) depicts the plot of the numerical solutions and the exact solution with $\varepsilon = 0.00001$ and $h = .01$

Here the numerical solutions and the exact solutions are approximately the same.

We now examine more systematically the behavior of the numerical solution of problem A, obtained through this method by further numerical experiments.

We compute the numerical solutions for the values $\varepsilon = 2^{-i}, i = 1,\ldots\ldots,18$ and values of N between 8 and 512, and we then tabulate the corresponding exact errors $E_\varepsilon^N$ and $E^N$ for these values of $\varepsilon$ and N.

From the Table-I we see that the maximum value of the error in each column is not less than 0.19800 and thus it is clear than this method is not $\varepsilon$-uniform at the mesh points for this problem.

In other words for any value of N there is always some value of $\varepsilon$ for which the error is at least 0.19800. Thus we conclude that the discussed method is not a practical method.

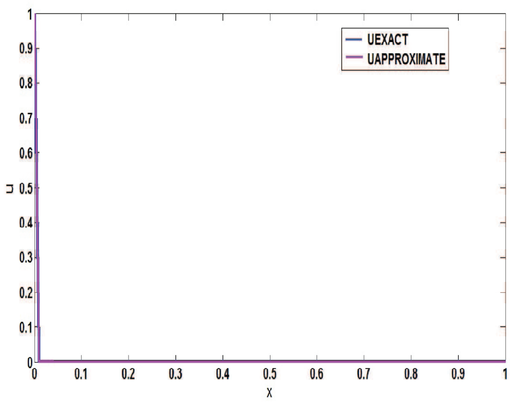


Fig 3(a)

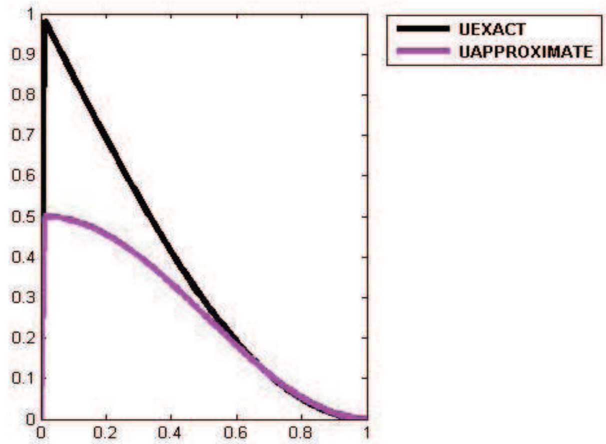


Fig 3(b)

| $\varepsilon$ | Table-I Number of Intervals N | | | | | | |
|---|---|---|---|---|---|---|---|
| | **8** | **32** | **16** | **64** | **128** | **256** | **512** |
| $2^{-1}$ | 0.06580 | 0.03641 | 0.0195 | 0.00993 | 0.00504 | 0.00254 | 0.00128 |
| $2^{-2}$ | 0.13037 | 0.07593 | 0.04146 | 0.02175 | 0.01115 | 0.00565 | 0.00284 |
| $2^{-3}$ | **0.19800** | 0.13211 | 0.07656 | 0.04172 | 0.02186 | 0.01121 | 0.00567 |
| $2^{-4}$ | 0.18168 | **0.19800** | 0.13212 | 0.07657 | 0.04172 | 0.02186 | 0.01121 |
| $2^{-5}$ | 0.11078 | 0.18168 | **0.19800** | 0.13212 | 0.07657 | 0.04172 | 0.02186 |
| $2^{-6}$ | 0.05882 | 0.11078 | 0.18168 | **0.19800** | 0.13212 | 0.07657 | 0.04172 |
| $2^{-7}$ | 0.03030 | 0.05882 | 0.11078 | 0.18168 | **0.19800** | 0.13212 | 0.07657 |
| $2^{-8}$ | 0.01538 | 0.03030 | 0.05882 | 0.11078 | 0.18168 | **0.19800** | 0.13212 |
| $2^{-9}$ | 0.00775 | 0.01538 | 0.03030 | 0.05882 | 0.11078 | 0.18168 | **0.19800** |
| $2^{-10}$ | 0.00389 | 0.00775 | 0.01538 | 0.03030 | 0.05882 | 0.11078 | 0.18168 |
| $2^{-12}$ | 0.00098 | 0.00195 | 0.00389 | 0.00775 | 0.01538 | 0.03030 | 0.05882 |
| $2^{-14}$ | 0.00024 | 0.00049 | 0.00098 | 0.00195 | 0.00389 | 0.00775 | 0.01538 |
| $2^{-16}$ | 6.10314e-005 | 0.00012 | 0.00024 | 0.00049 | 0.00098 | 0.00195 | 0.00389 |
| $2^{-18}$ | 1.52586e-005 | 3.05166e-005 | 6.10314e-005 | 0.00012 | 0.00024 | 0.00049 | 0.00098 |
| $E^N$ | 0.19800 | 0.19800 | 0.19800 | 0.19800 | 0.19800 | 0.19800 | 0.19800 |

**Piece-wise uniform fitted meshes:** We now introduce simple piecewise-uniform meshes of the form $= \overline{\Omega_\varepsilon^N} = \{x_i / x_i = 2i\sigma/N\, , i \le N/2\ ;\ x_i = x_i + 2i(1-\sigma)/N\, , N/2 < i\}$
Where the transition parameter $\sigma$ which determines the point of transition from a fine to a coarse mesh , is fitted to the boundary layer by taking$\sigma = \min\{0.5, 0.5\frac{\varepsilon \ln(N)}{\alpha}\}$
We call a piecewise-uniform mesh with this special choice of $\sigma$ a *piecewise-uniform* fitted mesh. This piecewise-uniform mesh is only slightly more complex than a uniform mesh, because it is simply two uniform meshes glued together at a carefully chosen transition point. It reduces to a uniform mesh if $\sigma$ = 0.5

**Upwind Finite Difference Method On Piecewise Uniform Fitted Mesh:** $\varepsilon\delta^2 u_\varepsilon(x_i) + a(x_i)D^+ u_\varepsilon(x_i) = f(x_i)\ ;\ x_i \epsilon \Omega^N$
where $\overline{\Omega^N}$ is the piecewise-uniform fitted mesh.
Consider the convection-diffusion problem A and B, Applying this Method and solving the resulting system of algebraic equations we get numerical solutions which lie close to the exact solution. The following Fig.4(a)&(b) depicts the plot of the exact solution and the numerical solution with $\alpha = 2, \varepsilon = 0.01$ and $N = 16$.

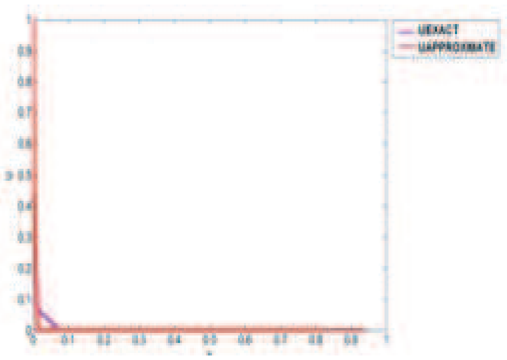

Fig.4(a)

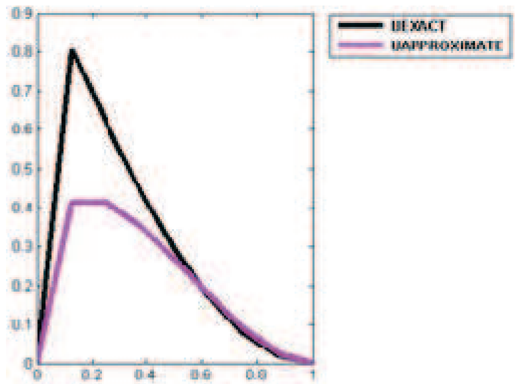


Fig.4(b)

| $\varepsilon$ | Table-II Number of Intervals N | | | | | | |
|---|---|---|---|---|---|---|---|
| | **8** | **32** | **16** | **64** | **128** | **256** | **512** |
| $2^{-2}$ | 0.26530 | 0.17547 | 0.11479 | 0.07364 | 0.04583 | 0.02718 | 0.01484 |
| $2^{-4}$ | 0.33194 | 0.23282 | 0.16327 | 0.11442 | 0.08010 | 0.05602 | 0.03913 |
| $2^{-6}$ | 0.34819 | 0.24579 | 0.17351 | 0.12247 | 0.08642 | 0.06098 | 0.04302 |
| $2^{-8}$ | 0.35222 | 0.24895 | 0.17597 | 0.12437 | 0.08790 | 0.06213 | 0.04391 |
| $2^{-10}$ | 0.35322 | 0.24974 | 0.17657 | 0.12484 | 0.08827 | 0.06241 | 0.04412 |
| $2^{-12}$ | 0.35347 | 0.24993 | 0.17673 | 0.12496 | 0.08836 | 0.06248 | 0.04418 |
| $2^{-14}$ | 0.35353 | 0.24998 | 0.17676 | 0.12499 | 0.08838 | 0.06249 | 0.04419 |
| $2^{-16}$ | 0.35355 | 0.25000 | 0.17677 | 0.12500 | 0.08839 | 0.06250 | 0.04419 |
| $2^{-18}$ | **0.35355** | **0.25000** | **0.17678** | **0.12500** | **0.08839** | **0.06250** | **0.04419** |
| □□ | 0.35355 | 0.25000 | 0.17678 | 0.12500 | 0.08839 | 0.06250 | 0.04419 |

Thus combining the finite difference method with this piecewise uniform mesh we get good approximations to the numerical solutions. In this method the piecewise uniform mesh always has mesh points in the boundary layer; in fact half of the mesh points are there, and so the boundary layer is resolved by this method.

We compute numerical solutions of problem A, for the values ε $=2^{-2i}$, i=1,…,9 and values of N between 8 and 512,and then tabulate the corresponding the exact errors $E_{\varepsilon}^{N}$ and $E^{N}$ for these values of ε and N.

Here for all values of ε the error decreases steadily with increasing N. From the Table-II, it is significant that the ε-uniform errors decreases rapidly as N increases. Thus the method is ε-uniform.

**Conclusion:** We conclude that the simple piecewise-uniform fitted mesh in conjunction with the standard upwind finite difference operator, yields a layer resolving $\varepsilon -$uniform method, also one can improve the approximate solution by applying a hybrid methods on non uniform mesh.

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