EXISTENCE OF A LATTICE NON MEASURABLE

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Abstract: This paper prescribes that the existence of lattice non measurable in [0, 1). For this we prove, if E is a lattice measurable and $\alpha \in$ [0, 1), then E + α is lattice measurable. Also m (E + α) = m (E) and if E is a lattice measurable, E $\leq$ [0, 1) and $\alpha \in$ [0, 1), then E $\oplus$ **α is** lattice **measurable and also m** (E $\oplus$ **α) = m(E).**
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Introduction: In this paper the definitions and foundations of the lattice theory are based on measure theory. Incidentally the Caratheodary outer measure is a direct generalization of the Borel - Lebesgue outer measure is a well known form of Real Analysis. Hence the lattice measure m on the σ – algebra σ (L) have the following properties. (1) For an interval J, m (J) = l (J) (2) If { E_n } be a sequence of disjoint sets (For which m is defined), m($\overset{\infty}{\underset{n=1}{\vee}} E_n$) = $\sum_{n=1}^{\infty} m(E_n)$ (3) m is translation invariant, that is, if E is a set for which m is defined and if E + α is the lattice
{x + α | x $\in$ E} then m(E + α) = m(E)[1].
By Szasz [3], we know that every {a} is a sublattice of L. Furthermore L is also a sublattice of itself. The sublattices of L other than itself are called proper sublattices of L. That is the proper sublattices consisting of more than one element are called non trivial sublattices.
If a lattice is allowed in [0, 1) we get lattices that some are measurable and some are not measurable. Due to this the concept of lattice non measurable arises. Hereafter we called a lattice non measurable by a Gamma (Γ) – lattice and denote it as Γ - lattice. We construct a Γ - lattice by using the definition of variety of lattices. A variety is trivial if and only if it contains one - element lattices only. In section 2, is devoted to the concepts of translate of a lattice, variety of a lattice and the sum modulo1 of Mona Khare etrl[1], and the result[2] that, if E is a measurable set, then for every $\varepsilon > 0$, there is an open set O > E with m(O – E) < ε. Also we present the results[2] 3.1, 3.2, 3.3., 3.4. for further development. Further we show that if E is a lattice measurable and $\alpha \in$ [0, 1), then E + α is lattice measurable and also m (E + α) = m (E). As well as if E is a lattice measurable, E $\leq$ [0, 1) and $\alpha \in$ [0, 1), then E $\oplus$ α is lattice measurable and also m (E $\oplus$ α) = m(E). Finally we construct a lattice non measurablein [0, 1)

Preliminaries:

Definition2.1. Let E be a lattice measurable. Then the translate of a lattice E denoted by E + α and defined it as the set of all x + αsuch that x belongs to E and α belongs to [0, 1).

Definition2.2. If x and α are real numbers in [0, 1), we define Sum modulo 1 (or closed under $\oplus$) of x and α **to** be x $\oplus$ α = (x + α) ∧ 1 if x + α < 1or (x + α – 1) ∧ 1 if x + α $\geq$ 1.

Definition2.3. If a lattice measurable set E $\leq$ [0, 1) and α belongs to [0, 1). Then we define the lattice translate modulo 1 of E is denoted by E $\oplus$ α and defined as set of all x $\oplus$ α such that x belongs to E and α belongs to [0, 1).

Definition2.4. We know that every {a} is a sublattice of L. Furthermore L is also a sublattice of itself. The sublattices of L other than itself are called proper sublattices of L. That is the proper sublattices consisting of more than one element are called non trivial sublattices. Here we define a sublattice consisting only one element is called trivial sublattices and class of all trivial sublattices is called variety of lattice.

Note2.2.A variety is trivial if and only if it contains one - element lattices only.

Result2.1. If E is lattice measurable set then for every $\varepsilon > 0$ there is an open lattice O > E with m(O – E) < ε.

Result2.2. The difference relation ~ in [0, 1) is an equivalence relation.

Proof. Define ~ in [0, 1) as follows. For all x, y belongs to [0, 1) if x ~ y iff x – y is a rational number**.** Clearly x – x is a rational number. implies x ~ x implies ~ is reflexive.
Let x ~ y implies x – y is a rational number, y – x is also rational number implies y ~ x implies ~ is symmetric.
Let x ~ y and y ~ z implies x – y is a rational number and y – z is a rational number implies x – y + y – z = x – z is a rational number implies x ~ z implies ~ is transitive. Therefore ~ is equivalence.

Construction Of Lattice Non Measurable:

Result3.1.[2].If $E_1, E_2, \ldots\ldots$ are pair wise disjoint lattice measurable sets, and E = $\overset{\infty}{\underset{k=1}{\vee}} E_k$, then E is lattice measurable (or) Every σ - lattice is lattice measurable and also m(E) = $\sum_{k=1}^{\infty} m(E_k)$.

Result3.2.[2]. Suppose that $\{E_k\}$ is monotonic increasing sequence of lattice measurable sets and E = $\overset{\infty}{\underset{k=1}{\vee}} E_k$. Then m(E) = $\underset{n\to\infty}{Lt}$ m(E_n).

Result3.3.[2]. If $E_1, E_2, \ldots\ldots$ are lattice measurable **sets,**

then $\overset{\infty}{\underset{k=1}{\wedge}} E_k$ is lattice measurable (or) every δ- lattice is lattice measurable.

Result3.4[2]. Suppose that $\{E_k\}$ is a monotonic decreasing sequence of lattice measurable sets and E = $\overset{\infty}{\underset{k=1}{\wedge}} E_k$. Then m(E) = $\underset{n\to\infty}{Lt}\ m(E_n)$.

Result3.5.If $E_1, E_2, ...$ are pair wise disjoint lattice measurable sets and

E = $\overset{\infty}{\underset{k=1}{\vee}} E_k$ then m(E) = m($\overset{\infty}{\underset{k=1}{\vee}} E_k$) = $\sum_{k=1}^{\infty} m(E_k)$ = lim m(E_k).

Proof.Evidently this result is proved by using result3.1. and 3.2.

Lemma3.1. If E is a lattice measurable and **α**∈ [0, 1), then E + **α is** lattice measurable. Also prove that m(E + α) = m(E).

Proof: Let E be a lattice measurable and α∈ [0, 1).

By result2.1. E is a lattice measurable.

So there exists an open lattice O > E with m(O – E) < ε ….(1) .Since O is an open lattice, O + α is also an open lattice. clearly O + α > E + α.Which implies (O + α) – (E + α) = (O – E) + α (By Definition2.1.).We have m((O + α) – (E + α)) = m((O – E) + α) = m(O – E). (m is translation invariant) Hence m((O + α) – (E + α)) = m(O – E) < ε (From (1)). Therefore E + α is lattice measurable. Also evidently m(E + α) = m(E) (m is translation invariant).

Lemma3.2.If E be a lattice measurable,E ≤ [0, 1) and α∈ [0, 1) then E ⊕ α is lattice measurable also

m(E ⊕ α) = m(E).

Proof. Let E be a lattice measurable and E ≤[0, 1) and α ∈ [0, 1).Now [0, 1) = [0, 1 – α)∨ [1 – α, 1) and [0, 1 – α)∧ [1 – α, 1) = 0.Evidently E = E∧ [0, 1) = (E∧[0, 1 – α))∨ (E ∧ [1 – α, 1)).

Let E_1 = E∧[0, 1 – α) andE_2 = E ∧ [1 – α, 1). Clearly E =E_1∨E_2andE_1∧E_2 = ϕ. (Since E is lattice measurable) implies$E_1$and $E_2$are lattice measurable sets. By Result3.1. So m(E) = m(E_1∨E_2) = m(E_1) + m(E_2) ………(1). Since E = E_1∨E_2implies E⊕α = (E_1⊕α) ∨ (E_2⊕α) ………(2). NowE_1 ⊕ α = {x ⊕ α | x∈ E_1} = {(x + α)∧ 1 | x∈ E_1} = { x + α | x∈ E_2} = E_1+ α. AlsoE_2 ⊕ α = {x ⊕ α | x∈ E_2} = {(x + α– 1) ∧ 1| x∈ E_2 } { x + α - 1| x∈ E_2} = E_2 + (α – 1).By Lemma3.1.E_1 + α andE_2 + (α – 1) arelattice measurable. Which implies m(E_1⊕α) and m(E_2⊕α) are lattice measurable. Hence m(E ⊕ α) is lattice measurable (From (2) and Definition 2.3.).Also m(E_1 ⊕α) = m(E_1+α) = m(E_1) and m(E_2⊕ α) = m(E_2+ (α – 1)) = m(E_2) (Since m is translation invariant). Also we see that (E_1⊕ α) ∧(E_2⊕ α) = ϕ.Let x ∈ (E_1⊕α) ∧(E_2⊕α). Which implies x = a⊕α = b⊕α where a∈ E_1andb∈ E_2. Thena + α = a⊕α = b⊕α = b+ α – 1 Which leads toa + α = b + α – 1 Henceb - a= 1 a contradiction to the fact b - a< 1. Therefore(E_1⊕ α)∧(E_2⊕α) =ϕ. So by (2) m(E ⊕α) = m(E_1⊕α) + m(E_2⊕α) = m(E_1) + m(E_2) = m(E) (From (1)).

Theorem3.1. There is a lattice non measurable in [0, 1).

Proof: Define a relation ~ (difference relation) on [0, 1). Clearly this difference relation is an equivalence relation (From Result2.2.). Now this equivalence relation partitions [0, 1) into disjoint equivalent classes so that the difference between any two elements present in a same equivalent class is a rational number and the difference between any two elements from different equivalent classes is an irrational number. By variety of lattice(Definition2.4.) let P be a lattice which contains exactly one – element from each of the equivalent classes. That is P ≤ [0, 1) …… (1). Since all the rational numbers present in [0, 1) are countably infinite, they can be written in the form of a sequence $\{r_n\}_{n=0}^{\infty}$ with r_0 = 0 and $r_n \neq r_m$ for n ≠ m. Define P_n = P ⊕ r_n, n = 0, 1, 2,etc ……..(2).

NowP_0 = P⊕ r_0 = P. We claim that $P_n \wedge P_m = \phi$ for n ≠ m and $\overset{\infty}{\underset{n=0}{\vee}} P_n$ = [0, 1).

Part (1). To prove that $P_n \wedge P_m = \phi$ for n ≠ m. Let x ∈ $P_n \wedge P_m$, where m and n are non negative integers. Which implies x = $a \oplus r_n = b \oplus r_n$ where$a$, b∈ P. By Lemma3.2.$a + r_n = b + r_n$.

Then$a - b = r_n - r_m$ = rational number. So by our supposition $a$and $b$belongs to the same equivalent class. Then$a = b$. Hence $r_n = r_m$for n ≠ m. Which is a contradiction since all rational numbers are distinct. Therefore P_n's are pair wise disjoint.

Part(2).To prove that $\overset{\infty}{\underset{n=0}{\vee}} P_n$ = [0, 1).

Case(1). P ≤ [0, 1)(From (1)) and r_n∈ [0, 1). ThenP ⊕ r_n≤[0, 1).Which impliesP_n≤ [0, 1) (From (2)). Hence $\overset{\infty}{\underset{n=0}{\vee}} P_n$≤ [0, 1) …..(3).

Case(2). Let x be any arbitrary element in [0, 1). Which implies x must be any one of the equivalent class.

Let y belongs to P be an element in which x is present. Then x ~ y is a rational number. Which implies x ~ y = r_n. Which leads to x = y + r_n = y⊕ r_n belongs to P⊕ r_n = P_n (Since y ∈ P)

Then x ∈ $\overset{\infty}{\underset{n=0}{\vee}} P_n$. Hence [0, 1) ≤ $\overset{\infty}{\underset{n=0}{\vee}} P_n$ …..(4). By (3) and (4) We have $\overset{\infty}{\underset{n=0}{\vee}} P_n$ = [0, 1).

Part(3). We show that P is a Γ - lattice in [0, 1). Suppose P is lattice measurable. By Lemma3.2. EachP_n is lattice measurable and m(P) = m(P_n) for all n. Since [0, 1) =

$\overset{\infty}{\underset{n=0}{\vee}} P_n$ and P_n's are pair wise disjoint, we have 1 = m([0, 1)) = m($\overset{\infty}{\underset{n=0}{\vee}} P_n$) = $\sum_{n=0}^{\infty} m(P_n)$ = $\sum_{n=0}^{\infty} m(P)$,(By Result3.5.). Here either m(P) = 0 or m(P) > 0.

If m(P) = 0 then 1 = $\sum_{n=0}^{\infty} m(P)$ = 0, a contradiction.

If m(P) > 0 then 1 = $\sum_{n=0}^{\infty} m(P)$ = ∞, a contradiction.

Therefore P is a lattice non measurable set. Thus There is a lattice non measurable in [0, 1).

Conclusion: In this paper we established the exsistance of lattice non measurable in [0, 1). For this we proved, if E is a lattice measurable and α ∈ [0, 1), then E + α is lattice measurable. Also m (E + α) = m (E) and if E is a lattice measurable, E ≤ [0, 1) and α ∈ [0, 1), then E ⊕ **α is** lattice measurable and also m (E ⊕ α) = m(E).

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