GLOBAL ATTRACTIVITY OF A $(k+1)^{TH}$ ORDER NONLINEAR DIFFERENCE EQUATION

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Abstract: In this paper,we investigate the local stability and global attractivity of the positive solutions of the $(k+1)$ th order nonlinear difference equation, $x_{n+1}=\frac{\alpha+\beta x_n}{\gamma-x_{n-k}}$ $n = 0,1,2,3\ldots\ldots$ (1)
where $\alpha \geq 0$, $\gamma > \beta > 0$ are real numbers and $k \geq 1$ is an integer, and the initial conditions $x_{-k},\ldots,x_{-1}$ and x_0 are arbitrary. The positive equilibrium of (1) is a global attractor with a basin that depends on certain conditions imposed on higher coefficients. The feasibility and efficiency of the proposed techniques are verified by numerical examples and simulation results through MATLAB.

Keywords: Attractivity, Equilibrium, Local stability, Non linear difference equation.
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Introduction: Consider (1) , we prove that the positive equilibrium $\bar{x}$ of (1) is a global attractor with a basin that depends on certain conditions of the coefficients. Let I be any interval of real numbers and F be a continuous function defined on I^{k+1}.
Then, for initial conditions $x_{-k},\ldots x_{-1},\ x_0 \in I$, consider the difference equation,

$x_{n+1} = F(x_n, x_{n-1}, \ldots x_{n-k})$
$n = 0,1,2,3\ldots$ (2)

where $k \in \{1, 2, \ldots\}$ is an integer, and the function $F(u_0, u_1, \ldots u_k)$ has continuous partial derivatives. Equation (2) has a unique solution $\{x_n\}$,
where $n = -k$ to ∞.
A point $\bar{x}$ is called an equilibrium of (2) if
$\bar{x} = F(\bar{x}, \bar{x}, \ldots.\bar{x})$. That is, $x_n = \bar{x}$ for $n \geq 0$.
The linearized equation associated with (2) about an equilibrium $\bar{x}$ is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F}{\partial u_i}(\bar{x}, \bar{x}, \ldots.\bar{x}) y_{n-i}$$
$n = 0,1,2,3\ldots$ (3)

And its characteristic equation is [5],[6],[8],[9],

$$\lambda^{k+1} = \sum_{i=0}^{k} \frac{\partial F}{\partial u_i}(\bar{x}, \bar{x}, \ldots.\bar{x}) \lambda^{k-i} \quad (4)$$

This paper is organized as follows: Section 2 describes the preliminaries.Local stability and Global attractivity of (1) are shown in Section 3. Section 4 provides the numerical examples and simulation results. Section 5 concludes the paper.

Preliminaries:

Definition 2.1:

1) The equilibrium point $\bar{x}$ of (2) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k},\ldots x_{-1}$,$x_0 \in I$ with $\sum_{i=-k}^{0}|x_i - \bar{x}| < \delta$, we have $|x_n - \bar{x}| < \epsilon$ for all $n \geq -k$.
2) The equilibrium point $\bar{x}$ of (2) is called a global attractor if $x_{-k}, x_{-(k-1)}, \ldots x_0 \in I$ always implies $lim_{n\to\infty} x_n = \bar{x}$.
3) The equilibrium point $\bar{x}$ of (2) is called unstable if it is not locally stable.

Theorem 2.2: Assume that F is a C^1 function and let $\bar{x}$ be an equilibrium point of (2). Then the following statements are true:
i) If all the roots of the polynomial equation

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial F}{\partial u_i}(\bar{x}, \bar{x}, \ldots.\bar{x}) \lambda^{k-i} = 0 \quad (5)$$

lie in an open unit disk $|\lambda| < 1$, then the equilibrium $\bar{x}$ of (2) is asymptotically stable.
ii) If atleast one root of (5) has an absolute value greater than one, then the equilibrium $\bar{x}$ of (2) is unstable [5],[6],[8],[9].

3.Local Stability and Global Attractivity:

Lemma 3.1

Assume that p , $q \in$ R and $k \in \{0, 1, \ldots\}$. Then
$|p|+|q|<1$ (6)
is a sufficient condition for the asymptotic stability of the difference equation, $x_{n+1} + px_n + qx_{n-k} = 0$.
$n = 0,1,2,3...$ (7)
Now, let us consider the rational recursive sequence,

$$x_{n+1}=\frac{\alpha+\beta x_n}{\gamma-x_{n-k}}$$
$n = 0,1,2,3\ldots\ldots$ (8)

Where
$\alpha > 0, \gamma > \beta > 0$, $k \in \{1,2,3,...\}$. (9)
If (9) holds and
$\alpha<(\gamma-\beta)^2/4$ (10)
then (1) has two positive equilibria,

$$\bar{x}_1=\frac{(\gamma-\beta)+\sqrt{(\gamma-\beta)^2-4\alpha}}{2} \quad (11)$$

$$\bar{x}_2=\frac{(\gamma-\beta)-\sqrt{(\gamma-\beta)^2-4\alpha}}{2} \quad (12)$$

The linearized equation of (8) about the equilibrium $\bar{x}_i$ is

$$y_{n+1} - \frac{\beta}{\gamma-\bar{x}_i} y_n - \frac{\bar{x}_i}{\gamma-\bar{x}_i} y_{n-k} = 0$$
$i = 0,1,2\ldots$, $n = 0,1,2,3\ldots.$ (13)

The characteristic equation associated with (13) about $\bar{x}_1$ is

$$\lambda^{k+1} - \frac{2\beta}{(\gamma+\beta)-\sqrt{(\gamma-\beta)^2-4\alpha}} \lambda^k - \frac{(\gamma-\beta)+\sqrt{(\gamma-\beta)^2-4\alpha}}{(\gamma+\beta)-\sqrt{(\gamma-\beta)^2-4\alpha}} = 0 \quad (14)$$

In the view of

$$\left|\frac{(\gamma-\beta)+\sqrt{(\gamma-\beta)^2-4\alpha}}{(\gamma+\beta)-\sqrt{(\gamma-\beta)^2-4\alpha}}\right| > 1 \quad (15)$$

the equilibrium point $\bar{x}_1$ of (8) is unstable.
For the positive equilibrium $\bar{x}_2$, in the view of condition (9) , we have

$\bar{x}_2 = \frac{(\gamma-\beta)-\sqrt{(\gamma-\beta)^2-4\alpha}}{2} < \frac{(\gamma-\beta)}{2} < \gamma$ (16)

Consider,

$$|p| + |q| = \frac{\beta}{\gamma-\bar{x}_2} + \frac{\bar{x}_2}{\gamma-\bar{x}_2} = \frac{\beta+\bar{x}_2}{\gamma-\bar{x}_2}$$

$$= \frac{\beta+\frac{(\gamma-\beta)-\sqrt{(\gamma-\beta)^2-4\alpha}}{2}}{\gamma-\frac{(\gamma-\beta)-\sqrt{(\gamma-\beta)^2-4\alpha}}{2}}$$

$$= \frac{(\gamma+\beta)-\sqrt{(\gamma-\beta)^2-4\alpha}}{(\gamma+\beta)+\sqrt{(\gamma-\beta)^2-4\alpha}} < 1 \quad (17)$$

which by Lemma 3.1, implies that $\bar{x}_2$ is locally asymptotically stable [7],[8].

Now we show that the positive equilibrium $\bar{x}$ of (1) is a global attractor with a basin that depends on certain conditions imposed on the coefficients.

We consider the case α > 0.

Theorem 3.2

Consider the difference equation,

$x_{n+1} = g(x_n, x_{n-k})$

$n = 0,1,2,\ldots$ (18)

where $k \in \{1, 2, \ldots\}$, $g \in C[(0,\infty),(0,\infty)]$ is increasing in each of its arguments and the initial conditions $x_{-k}, \ldots x_{-1}, x_0$ are positive. Assume that (18) has a unique positive equilibrium $\bar{x}$ and that the function h is defined by,

$h(x) = g(x,x)$, $x \in (0,\infty)$ (19)

satisfies

$(h(x) - x)(x - \bar{x}) < 0$

for $x \neq \bar{x}$ (20)

Then $\bar{x}$ is a global attractor of all the positive solutions of (18).

Lemma 3.3

Assume that (9) and (10) holds. Then the following statements are true:

(i) $F(x,x)$ is a strictly increasing function in $(-\infty,\gamma)$.

(ii)If $(u_0,u_k) \in [-\alpha/\beta,\gamma) \times (-\infty,\gamma)$. Then F is a strictly increasing function in each of its arguments.

Lemma 3.4: Assume that (9) and (10) holds.

1) $0 < \bar{x}_2 < \bar{x}_1 < \gamma$ (21)

2) Suppose that the function h is defined by

$h(x) = F(x,x)$, $x \in (0,\bar{x}_1)$ (22)

Then $(h(x) - x)(x - \bar{x}_2) < 0$ for $x \neq \bar{x}_2$.

Proof : 1)Since $\bar{x}_1 - \gamma = \frac{(\gamma-\beta)+\sqrt{(\gamma-\beta)^2-4\alpha}}{2} - \gamma$

$= \frac{-\gamma-\beta+\sqrt{(\gamma-\beta)^2-4\alpha}}{2} < 0$

Therefore ,we have $\bar{x}_1 < \gamma$

By the definitions of $\bar{x}_1$ and $\bar{x}_2$, we have

$0 < \bar{x}_2 < \bar{x}_1 < \gamma$.

The proof is complete.

2) By the definition of h, we have

$$(h(x) - x)(x - \bar{x}_2) = \left(\frac{\alpha+\beta x}{\gamma-x} - x\right)(x - \bar{x}_2)$$

$$= \left(\frac{\alpha+\beta x-\gamma x+x^2}{\gamma-x}\right)(x - \bar{x}_2)$$

$$= \left(\frac{x^2-(\gamma-\beta)x+\alpha}{\gamma-x}\right)(x - \bar{x}_2)$$

$$= \frac{1}{\gamma-x}\left(x - \frac{(\gamma-\beta)+\sqrt{(\gamma-\beta)^2-4\alpha}}{2}\right) \times \left(x - \frac{(\gamma-\beta)-\sqrt{(\gamma-\beta)^2-4\alpha}}{2}\right)(x - \bar{x}_2)$$

$= \frac{1}{\gamma-x}(x - \bar{x}_1)(x - \bar{x}_2)^2 < 0$ for $x \neq \bar{x}_2$.

The proof is complete.

Theorem 3.5: Assume that (9) and (10) holds. Let $\{x_n\}$ be any solution of (1). If $(x_{-k}, \ldots x_{-1}, x_0) \in (-\infty,\bar{x}_1]^k \times [-\alpha/\beta,\bar{x}_1]$, then $0 \leq x_n \leq \bar{x}_1$, for $n \geq 1$.

Proof: By part (2) of Lemma 3.4, we have

$0 = F(-\alpha/\beta, x_{-k}) \leq x_1$

$= F(x_0, x_{-k}) \leq F(\bar{x}_1, \bar{x}_1) = \bar{x}_1$, (23) and

$0 < F(0, x_{-k+1}) \leq x_2$

$= F(x_1, x_{-k+1}) \leq F(\bar{x}_1, \bar{x}_1) = \bar{x}_1$. (24)

Hence, the result follows by induction. Therefore the proof is complete.

Theorem 3.6: Assume that (9) and (10) holds. Then the positive equilibrium $\bar{x}_2$ of (1) is a global attractor with a basin $S = (0,\bar{x}_1)^{k+1}$.

Proof: Let $\{x_n\}$ be a solution of (1) with the initial conditions $(x_{-k}, \ldots x_{-1}, x_0) \in S$. Then,by part (2) of Lemma 3.4 and Theorem 3.5, the function $F(u_0,u_k) \in C[(0,\bar{x}_1),(0,\bar{x}_1)]$ is a strictly increasing function in each of its arguments and the initial conditions $x_{-k}, \ldots x_{-1}, x_0$ are positive. Let the function h be defined by

$h(x) = F(x,x)$, $x \in (0,\bar{x}_1)$ (25)

Then by part (2) of Lemma 3.4, we have $(h(x) - x)(x - \bar{x}_2)$

$= \frac{1}{\gamma-x}(x - \bar{x}_1)(x - \bar{x}_2)^2 < 0$, $x \neq \bar{x}_2$ (26)

By Theorem 3.2, $\bar{x}_2$ is a global attractor of all the positive solutions of (1). Then,

$lim_{n\to\infty}\, x_n = \bar{x}_2$ (27)

The proof is complete.

Theorem 3.7: Assume that (9) and (10) holds. Then the positive equilibrium $\bar{x}_2$ of (1) is a global attractor with a basin $S = (-\infty,\bar{x}_1]^k \times [-\alpha/\beta,\bar{x}_1] \setminus \{(\bar{x}_1, \ldots \bar{x}_1)\}$.

Proof: Let $\{x_n\}$ be a solution of (1) with initial conditions $(x_{-k}, \ldots x_{-1}, x_0) \in S$. Then by Theorem 3.5, we have $x_n \in (0,\bar{x}_1)$,

for $n \geq 2$.

Hence, Theorem 3.6 implies that $lim_{n\to\infty}\, x_n = \bar{x}_2$

The proof is complete.

Now, we consider the case α = 0.

We find the asymptotic stability of the difference equation,

$x_{n+1} = \frac{\alpha+\beta x_n}{\gamma-x_{n-k}}$, $n = 0, 1, \ldots,$ (28)

where $\beta, \gamma > 0$, and $k \geq 1$.

By putting $x_n = \beta y_n$, (28 yields $y_{n+1} = \frac{y_n}{A-y_{n-k}}$,$n = 0,1, \ldots,$ (29)

where $A = \gamma/\beta$. Equation (29) has two equilibrium points $\bar{y}_1 = 0$ and $\bar{y}_2 = A-1$.

The linearized equation associated with (29) about $\bar{y}_i$

$z_{n+1} - \frac{1}{A-\bar{y}_i} z_n - \frac{\bar{y}_i}{A-\bar{y}_i} z_{n-k} = 0$

$i = 1,2$ and $n = 0,1,2,\ldots$ (30)

The characteristic equation of (30) about $\bar{y}_1$ is

$\lambda^{k+1} - \frac{1}{A}\lambda^k = 0$ (31)

This equation has two roots.

λ_1=0 and λ_2=$\frac{1}{A}$ (32)

Hence, by theorem 2.2

(1) if $\gamma > \beta$ then $\bar{y}_1$ is asymptotically stable.

(2) if $\gamma < \beta$ then $\bar{y}_1$ is a saddle point and

(3) if $\gamma = \beta$ then linearized stability analysis fails.

we assume that $\gamma > \beta$ which implies, $A > 1$.

Lemma 3.8: Assume that the initial conditions $(y_{-k}, \dots . y_{-1}, y_0) \in [0, \bar{y}_2]^{k+1} \backslash \{(\bar{y}_2, \dots \bar{y}_2)\}$. Then every solution $\{y_n\}$ of (29) is nonnegative and monotonically decreases to zero.

Proof: Since, $(y_{-k}, \dots . y_{-1}, y_0) \in [0, \bar{y}_2]^{k+1} \backslash \{(\bar{y}_2, \dots \bar{y}_2)\}$, we have

$$0 \le y_1 = \frac{y_0}{A - y_{-k}} \le \frac{\bar{y}_2}{A - \bar{y}_2} = \bar{y}_2 \qquad (33)$$

and

$$0 \le y_2 = \frac{y_1}{A - y_{1-k}} \le \frac{\bar{y}_2}{A - \bar{y}_2} = \bar{y}_2. \qquad (34)$$

By induction, we can see that $0 \le y_n \le \bar{y}_2$, for $n \ge -k$. Since, for $n \ge 0$,

$$y_n - y_{n+1} = y_n - \frac{y_n}{A - y_{n-k}}$$

$$= \frac{y_n}{A - y_{n-k}}(A - 1 - y_{n-k}) \ge 0 \qquad (35)$$

Then , $y_n \ge y_{n+1}$ for $n \ge 0$.

Hence, $\{y_n\}$ is a non negative and monotonically decreasing sequence.

Set $lim_{n\to\infty} y_n = l > 0$

By taking limits on both sides of (29), we have $l = \frac{l}{A - l}$

Then, $l = 0$ or $l = A - 1$.

If $l = A - 1 = \bar{y}_2$, then by the monotonic character of sequence $\{y_n\}$, we have $y_n = \bar{y}_2$,for all $n \ge 0$. Hence $y_{-k} = \dots = y_{-1} = \bar{y}_2$. This is a contradiction. So, $l = 0$. The proof is complete.

Corollary 3.9: The equilibrium $\bar{y}_1 = 0$ of (29) is a global attractor with a basin

$S = [0, \bar{y}_2]^{k+1} \backslash \{(\bar{y}_2, \dots \bar{y}_2)\}$.

Theorem 3.10: The equilibrium $\bar{y}_1 = 0$ of (29) is a global attractor with a basin $S = (-\infty, \bar{y}_2]^k \times [0, \bar{y}_2] \backslash \{(\bar{y}_2, \dots \bar{y}_2)\}$.

Proof: Assume $(y_{-k}, \dots . . y_{-1}, y_0) \in S$, then we have

$$0 = \frac{0}{A - y_{-k}} \le y_1 = \frac{y_0}{A - y_{-k}}$$

$$< \frac{\bar{y}_2}{A - \bar{y}_2} = \bar{y}_2$$

$$0 \le y_2 = \frac{y_1}{A - y_{1-k}} < \frac{\bar{y}_2}{A - \bar{y}_2} = \bar{y}_2$$

The result now follows from Corollary 3.9 and the proof is complete[7],[8].

Simulation Results: To enrich our results of this section, we consider the following numerical examples which represents different types of solutions to (1).

Example 4.1: Consider, $x_{-5} = 1$, $x_{-4} = 2$, $x_{-3} = 4$, $x_{-2} = -1$, $x_{-1} = -2$, $x_0 = 7$, $\alpha = 1, \beta = -2$, and $\gamma = 4$

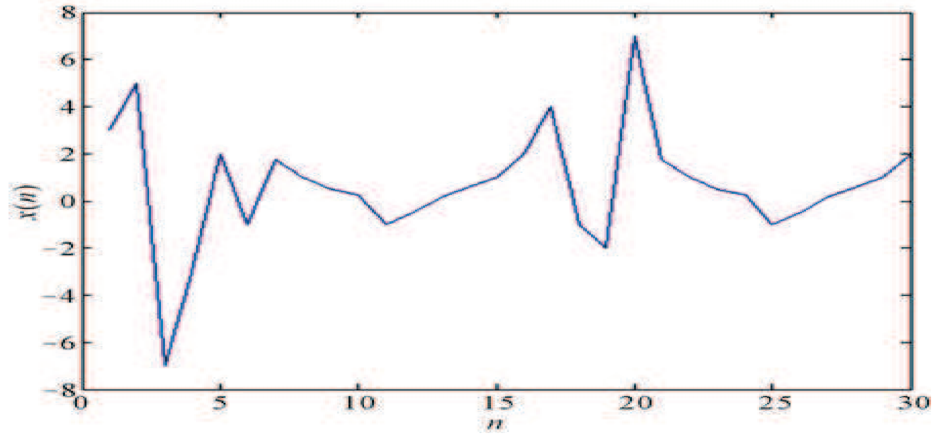

Fig 1: Plot of Example 4.1 of (1)

Example 4.2: Consider $x_{-5} = 1$, $x_{-4} = -1$, $x_{-3} = 2$, $x_{-2} = -6$, $x_{-1} = 5$, $x_0 = -4$, , $\alpha = 3$, $\beta = 1$, and $\gamma = -1$

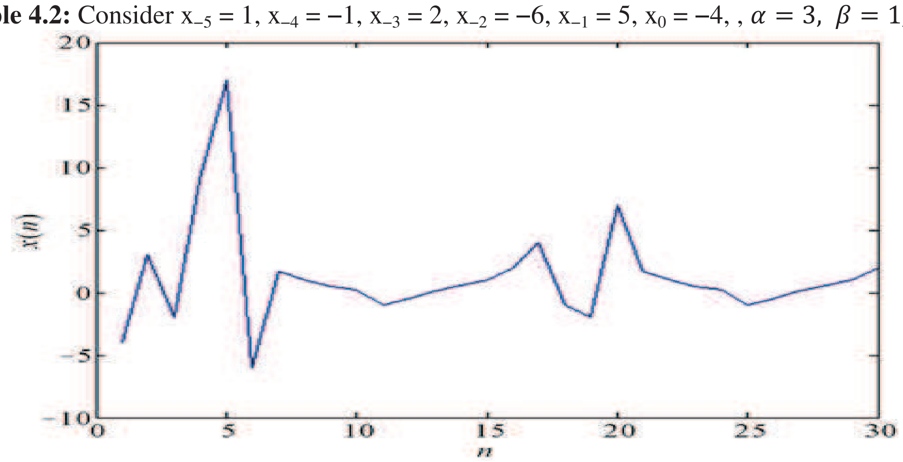

Fig 2: Plot of Example 4.2 of (1)

Conclusion: In this paper ,we have investigated the local stability and global attractivity of a nonlinear $(K+1)^{th}$ order difference equation,

$$x_{n+1} = \frac{\alpha+\beta x_n}{\gamma - x_{n-k}} \qquad n=0,1,2,3\ldots$$

We arrived the following as a certain key features:

1. The equilibrium point $\bar{x}_1$ of (1) is unstable whereas $\bar{x}_2$ is locally asymptotically stable.
2. The positive equilibrium $\bar{x}_2$ of (1) is a global attractor with the basins
   i. $S = (0, \bar{x}_1)^{k+1}$ and
   ii. $S=(-\infty, \bar{x}_1]^k \times [-\alpha/\beta, \bar{x}_1]$
3. The equilibrium $\bar{y}_1 = 0$ of (29) is a global attractor with the basins $S = [0, \bar{y}_2]^{k+1} \backslash \{(\bar{y}_2, \ldots \bar{y}_2)\}$ and $S = (-\infty, \bar{y}_2]^k \times [0, \bar{y}_2] \backslash \{(\bar{y}_2, \ldots \bar{y}_2)\}$.

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