EDGE CHROMATIC Δ - CRITICAL GRAPHS

K. KAYATHRI, J. SAKILA DEVI

Abstract: In this paper, we study the size of edge chromatic Δ-critical graphs in several classes of critical graphs. We have obtained bounds on size of Δ-critical graphs with $\delta = \Delta - 1$ which are stronger than the Vizing's Weaker Conjectured bound. Also in this paper, we give a bound on maximum degree Δ of Δ-critical graphs of odd order with $\delta = \Delta - 2$ in terms of their minor vertices through which we can produce a list of impossible degree sequences for Δ-critical graphs of any odd order. In this paper, we list out for n =15.

Keywords: class one, class two, edge colouring, edge chromatic critical graphs.

Introduction: Throughout this paper, Let G = (V,E) be a simple graph with n vertices and m edges. Δ(G) , δ (G) denote the maximum degree and the minimum degree of G. The number of vertices of degree j is denoted by n_j. $\pi(G) = 1^{n_1} 2^{n_2} \cdots \Delta^{n_\Delta}$ denotes the degree sequence of G, where if n_j = 0, then the factor j^{n_j} is customarily omitted in $\pi(G)$. The chromatic index χ'(G) of a graph G is the minimum number of colours required to colour the edges of G so that no two adjacent edges receive the same colour. A famous theorem of Vizing[13] states that the chromatic index χ'(G) of a simple graph G is Δ(G) or Δ(G)+1, where Δ(G) denotes the maximum vertex degree in G. A graph G is of class one if χ'(G) = Δ(G) and is of class two otherwise. A class two graph G is (chromatic index) critical if χ'(G - e) < χ'(G) for each edge e of G. If we want to stress the maximum vertex degree of a critical graph G, we say G is Δ-critical.

Known Results: A well-known fundamental result, known as Vizing's Adjacency Lemma(VAL) describes an important property of Δ-critical graphs. We will be repeatedly using this result.

R1: Vizing's Adjacency Lemma(VAL)[11]: In a Δ-critical graph G, if vw is an edge and $d(v) = k$, then w is adjacent with at least $\Delta - k + 1$ other vertices of degree Δ. In particular, G has at least $\Delta - \delta + 2$ vertices of degree Δ.

R2:Vizing's Conjecture [12]: If G is a Δ-critical graph with n vertices, m edges and $\Delta \geq 3$, then $m \geq \frac{1}{2}\left(n(\Delta - 1) + 3\right)$.

This conjecture has been verified for the graphs with $\Delta \leq 6$. [5,7,8,10]

R3:Vizing's Weaker Conjecture[12]: Recognizing that Vizing's Conjecture is probably difficult to settle, Vizing remarks that he is unable to settle the following simple problem:

Is it true that if G is simple and Δ-critical,then $m \geq \frac{\Delta^2}{2}$?

This problem was referred as Vizing's Weaker Conjecture.

In this paper, we have improved Vizing's Weaker Conjectured bound in several classes of critical graphs.

The following known theorem gives bounds on size of edge chromatic critical graphs of odd and even orders.

R4: Theorem[14]: If G is a Δ-critical graph of order n having the minimum valency δ, then

(i) $m \leq \frac{1}{2}(n-1)\Delta + 1$ if n is odd

(ii) $m \leq \frac{1}{2}(n-2)\Delta + \delta - 1$ if n is even

R5: Theorem[9]: Let G be a Δ-critical graph with $n_\Delta = (\Delta - \delta + 2) + l$ where $l \geq 0$. If $\Delta \geq (\delta - 1 - l)(\delta - 2 - l)$, then $2m \geq \Delta^2$.

Size of Δ-critical graphs with $\delta = \Delta - 1$

In this section, we have obtained bounds on size of Δ-critical graphs with $\delta = \Delta - 1$ which are stronger than Vizing's Weaker Conjectured bound.

Theorem 3.1: Let G be a Δ-critical graph with $\Delta \geq 3$, $\delta = \Delta - 1$ and n odd. Then $m \geq \frac{\Delta^2}{2} + 1$.

Proof: By R4, $m \leq \frac{1}{2}(n-1)\Delta + 1$

$$\Rightarrow \frac{\delta n_\delta + \Delta n_\Delta}{2} \leq \frac{1}{2}(n-1)\Delta + 1$$
$$\Rightarrow \delta n_\delta + \Delta n_\Delta \leq (n-1)\Delta + 2$$
$$\Rightarrow (\Delta - 1)(n - n_\Delta) + \Delta n_\Delta \leq n\Delta - \Delta + 2$$
$$\Rightarrow n\Delta - n - \Delta n_\Delta + n_\Delta + \Delta n_\Delta \leq n\Delta - \Delta + 2$$
$$\Rightarrow n_\Delta \leq n - \Delta + 2.$$

Since $n - n_\Delta = n_\delta, n_\delta \geq \Delta - 2$.

By VAL, $n_\Delta \geq \Delta - \delta + 2$.

Hence $2m = \delta n_\delta + \Delta n_\Delta$

$$\geq (\Delta-1)(\Delta-2)+\Delta(\Delta-\delta+2)$$
$$=\Delta^2-3\Delta+2+\Delta(\Delta-\Delta+1+2)=\Delta^2+2$$

Thus $m \geq \frac{\Delta^2}{2}+1.$

Theorem 3.2: Let G be a Δ-critical graph with $\Delta \geq 3$, $\delta=\Delta-1$, n odd and $n_\Delta=\Delta-\delta+2+l.$

Then $m \geq \frac{\Delta^2}{2}+\frac{l\Delta}{2}+1.$

Proof: Proceeding as in theorem 3.1, we get that $n_\delta \geq \Delta-2.$ Hence

$$2m=\delta n_\delta+\Delta n_\Delta=(\Delta-1)n_\delta+\Delta(\Delta-\delta+2+l)$$
$$\geq (\Delta-1)(\Delta-2)+\Delta(\Delta-\Delta+1+2+l)$$
$$\Rightarrow 2m \geq \Delta^2-\Delta-2\Delta+2+3\Delta+l\Delta$$
$$\Rightarrow 2m \geq \Delta^2+l\Delta+2.$$

Hence $m \geq \frac{\Delta^2}{2}+\frac{l\Delta}{2}+1.$

Theorem 3.3: Let G be a Δ-critical graph with $\Delta \geq 3$, $\Delta=n-2$, $\delta=\Delta-1$, n odd.

Then $\pi(G)=(\Delta-1)^{n-3}\Delta^3$ with $m=\frac{n(\Delta-1)+3}{2}$ (or) $\pi(G)=(\Delta-1)^{n-4}\Delta^4$ with $m=\frac{n(\Delta-1)}{2}+2.$

Proof: Proceeding as in theorem 3.1, we get that $n_\delta \geq \Delta-2$.

Hence $n_\Delta=n-n_\delta \leq n-\Delta+2$
$$=\Delta+2-\Delta+2=4.$$

Also by VAL, $n_\Delta \geq 3.$ Thus $n_\Delta=3$ (or) 4.

Case(i): Let $n_\Delta=3$.

Then $\pi(G)=(\Delta-1)^{n-3}\Delta^3$.

Hence $2m=(n-3)(\Delta-1)+3\Delta$
$$=n(\Delta-1)-3\Delta+3+3\Delta$$
$$\Rightarrow m=\frac{n(\Delta-1)+3}{2}.$$

Case(ii): Let $n_\Delta=4$.

Then $\pi(G)=(\Delta-1)^{n-4}\Delta^4$.

Hence $2m=(n-4)(\Delta-1)+4\Delta$
$$=n(\Delta-1)-4\Delta+4+4\Delta$$
$$\Rightarrow m=\frac{n(\Delta-1)}{2}+2.$$

Remark: By theorem 3.3, Vizing's bound on the size is attained for the Δ-critical graphs with $\Delta \geq 3$, n = Δ+2, $\delta=\Delta-1$ and n odd.

Theorem 3.4: If G is a Δ-critical graph with $\Delta \geq 3$, $\delta=\Delta-1$, n odd and $n_\Delta \geq 2\Delta-\delta+2$, then $m \geq \Delta^2+1.$

Proof: Since $n_\Delta \geq 2\Delta-\delta+2,$
$n_\Delta=\Delta-\delta+2+l$ for some $l \geq \Delta.$

Hence by the theorem 3.2,
$$m \geq \frac{\Delta^2}{2}+\frac{l\Delta}{2}+1 \geq \frac{\Delta^2}{2}+\frac{\Delta^2}{2}+1.$$

Thus $m \geq \Delta^2+1.$

Theorem 3.5: Let G be a Δ-critical graph with $\Delta \geq 4$, $\Delta=n-1$, $\delta=\Delta-1$ and n odd. Then Δ is even and $\pi(G)=(\Delta-1)^{n-3}\Delta^3$.

Proof: Proceeding as in the theorem 3.1, we get that $n_\Delta \leq n-\Delta+2=3.$ By VAL, $n_\Delta \geq 3.$ Thus $n_\Delta=3.$ Hence $\pi(G)=(\Delta-1)^{n-3}\Delta^3$. Since n is odd, n_δ is even. Also n_Δ is odd.

Since the number of vertices of odd degree in a graph is even, Δ is even.

Corollary 3.6: Let G be a Δ-critical graph with $\Delta \geq 4$, $\Delta=n-1$, $\delta=\Delta-1$ and n odd.

Then $m=\frac{(\Delta-1)n+3}{2}.$

Proof: By theorem 3.5, we get that
$$\pi(G)=(\Delta-1)^{n-3}\Delta^3.$$

Hence $2m=(\Delta-1)(n-3)+3\Delta.$

Thus $m=\frac{(\Delta-1)n+3}{2}.$

Note that, in this case, we get that the size of Δ-critical graph is exactly equal to the Vizing's conjectured bound.

Theorem 3.7: Let G be a Δ-critical graph with $\Delta \geq 4$, $\Delta=n-1$, $\delta=\Delta-1$ and n even, then $n_\delta \geq 4.$ In addition, if $n_\Delta \geq 2\Delta-\delta+2$,

then $m \geq \frac{\Delta^2}{2}+\frac{7}{2}\Delta-2.$

Proof: By R4, $m \leq \frac{1}{2}(n-2)\Delta+\delta-1$
$$\Rightarrow \delta n_\delta+\Delta n_\Delta \leq (n-2)\Delta+2\delta-2$$
$$\Rightarrow (\Delta-1)(n-n_\Delta)+\Delta n_\Delta$$
$$\leq n\Delta-2\Delta+2\Delta-2-2$$
$$\Rightarrow n\Delta-n-\Delta n_\Delta+n_\Delta+\Delta n_\Delta \leq n\Delta-4.$$

Hence $n_\Delta \leq n-4.$ Since $n=n_\delta+n_\Delta$ and $n_\Delta \leq n-4,$ $n_\delta \geq 4.$

By VAL, $n_\Delta \geq \Delta-\delta+2.$

Thus $n_\Delta=\Delta-\delta+2+l,$ $l \geq 0.$
$$2m=\delta n_\delta+\Delta n_\Delta=(\Delta-1)n_\delta+\Delta(\Delta-\delta+2+l)$$
$$\geq 4(\Delta-1)+\Delta^2-\delta\Delta+2\Delta+l\Delta$$

$= 4\Delta - 4 + \Delta^2 - \Delta^2 + \Delta + 2\Delta + l\Delta$

$= 7\Delta + l\Delta - 4. \quad \Rightarrow m \geq \left(\frac{7+l}{2}\right)\Delta - 2.$

Since $n_\Delta \geq 2\Delta - \delta + 2$, $l \geq \Delta$.

Thus $m \geq \frac{\Delta^2}{2} + \frac{7}{2}\Delta - 2.$

Theorem 3.8: Let G be a Δ-critical graph where $\Delta \geq 3$, $\Delta = n - i$, $\delta = \Delta - 1$ and n odd. Then $\frac{(\Delta-1)n+3}{2} \leq m \leq \frac{\Delta^2 + (i-1)\Delta}{2} + 1$. [Note that Vizing's Conjecture is attained here.]

Proof: $2m = \delta n_\delta + \Delta n_\Delta$

$= (\Delta-1)n_\delta + \Delta(n - n_\delta) = \Delta n_\delta - n_\delta + n\Delta - \Delta n_\delta$

$= (\Delta-1)n + n - n_\delta = (\Delta-1)n + n_\Delta.$

Since $n_\Delta \geq 3$, $2m \geq (\Delta-1)n + 3.$

(i.e.) $m \geq \frac{(\Delta-1)n+3}{2}.$

Proceeding as in theorem 3.1, we get that $n_\Delta \leq n - \Delta + 2$ and hence $n_\Delta \leq i + 2.$

Thus $2m = (\Delta-1)n_{\Delta-1} + \Delta n_\Delta$

$= (\Delta-1)(n - n_\Delta) + \Delta n_\Delta = n\Delta - n - \Delta n_\Delta + n_\Delta + \Delta n_\Delta$

$\leq (\Delta + i)\Delta - \Delta - i + i + 2 = \Delta^2 + i\Delta - \Delta + 2.$

Hence $m \leq \frac{\Delta^2 + (i-1)\Delta}{2} + 1.$

Theorem 3.9: Let G be a Δ-critical graph with $\delta = 7$, $n \geq \Delta + 3$ and $n_\Delta \geq \Delta - 3$.

Then $2m \geq \Delta^2 - 3\Delta + 42.$ Also $2m \geq \Delta^2$ if $\Delta \geq 12$.

Proof: Since $n_\Delta \geq \Delta - 3$, $n \geq \Delta + 3$ and $2m \geq \Delta n_\Delta + (n - n_\Delta)\delta$, we have

$2m \geq (\Delta - \delta)n_\Delta + \delta n \geq (\Delta - 7)(\Delta - 3) + 7(\Delta + 3)$

$= \Delta^2 - 10\Delta + 21 + 7\Delta + 21 = \Delta^2 - 3\Delta + 42.$

By VAL, $n_\Delta \geq \Delta - \delta + 2.$

Let $n_\Delta = \Delta - \delta + 2 + l$ where $l \geq 0$.

Now $n_\Delta \geq \Delta - 3$ and $\delta = 7$

$\Rightarrow \Delta - 3 \leq \Delta - 7 + 2 + l \Rightarrow l \geq 2.$

Since $l \geq 2$, we get that $(\delta - 1 - l)(\delta - 2 - l) = (6 - l)(5 - l) \leq 4 \times 3 = 12.$

Hence by R5, $2m \geq \Delta^2$ if $\Delta \geq 12$.

Corollary 3.10:

(i) If G is a 8-critical graph with $\delta = 7$, $n \geq 11$ and $n_8 \geq 5$, then $2m \geq \Delta^2 + 18.$

(ii) If G is a 9-critical graph with $\delta = 7$, $n \geq 12$ and $n_9 \geq 6$, then $2m \geq \Delta^2 + 15.$

(iii) If G is a 10-critical graph with $\delta = 7$, $n \geq 13$ and $n_{10} \geq 7$, then $2m \geq \Delta^2 + 12.$

(iv) If G is a 11-critical graph with $\delta = 7$, $n \geq 14$ and $n_{11} \geq 8$, then $2m \geq \Delta^2 + 9.$

(v) If G is a 12-critical graph with $\delta = 7$, $n \geq 15$ and $n_{12} \geq 9$, then $2m \geq \Delta^2 + 6.$

(vi) If G is a 13-critical graph with $\delta = 7$, $n \geq 16$ and $n_{13} \geq 10$, then $2m \geq \Delta^2 + 3.$

Note that, in all the above cases, we have improved the Vizing's Weaker Conjectured bound.

4. Bounds on maximum degree for Δ -critical graphs with $\delta = \Delta - 2$

The following theorem gives a bound on maximum degree Δ of Δ-critical graphs of odd order with $\delta = \Delta - 2$ in terms of their minor vertices.

Theorem 4.1: If G is a Δ-critical graph of odd order with $\delta = \Delta - 2$, then $\Delta \leq n_{\Delta-1} + 2n_{\Delta-2} + 2.$

Proof:

$m = \frac{1}{2}\left[(\Delta-2)n_{\Delta-2} + (n - n_{\Delta-2} - n_\Delta)(\Delta-1) + \Delta n_\Delta\right] = \frac{1}{2}\left[(\Delta-1)n + n_\Delta - n_{\Delta-2}\right].$ (1)

Since n is odd, by R4, $m \leq \frac{1}{2}(n-1)\Delta + 1.$ From (1),

$\frac{1}{2}\left[(\Delta-1)n + n_\Delta - n_{\Delta-2}\right] \leq \frac{1}{2}(n-1)\Delta + 1.$

$\Rightarrow n\Delta - n + n_\Delta - n_{\Delta-2} \leq n\Delta - \Delta + 2$

$\Rightarrow \Delta \leq n - n_\Delta + n_{\Delta-2} + 2 \Rightarrow \Delta \leq n_{\Delta-1} + 2n_{\Delta-2} + 2.$

Remark 4.2: Using Theorem 4.1, we can list out some impossible degree sequences for a Δ–critical graph G with $\delta = \Delta - 2$ of any order. Since critical graphs of order $n \leq 14$ were considered by various persons [1,2,3,4,6,14], in this paper, we list out impossible degree sequences for $n = 15$.

(i) $n = 15$, Δ is odd and $\delta = \Delta - 2$:

Here $n_{\Delta-1}$ is odd.

n_Δ	$n_{\Delta-1}$	$n_{\Delta-2}$	$n_{\Delta-1}+2n_{\Delta-2}+2$	Impossible degree sequences
13	1	1	5	$(\Delta-2)(\Delta-1)\Delta^{13}$ for $\Delta \geq 7$
12	1	2	7	$(\Delta-2)^2(\Delta-1)\Delta^{12}$ for $\Delta \geq 9$
11	3	1	7	$(\Delta-2)(\Delta-1)^3\Delta^{11}$ for $\Delta \geq 9$
11	1	3	9	$(\Delta-2)^3(\Delta-1)\Delta^{11}$ for $\Delta \geq 11$
10	3	2	9	$(\Delta-2)^2(\Delta-1)^3\Delta^{10}$ for $\Delta \geq 11$
10	1	4	11	$11^4 12\, 13^{10}$
9	5	1	9	$(\Delta-2)(\Delta-1)^5\Delta^9$ for $\Delta \geq 11$
9	3	3	11	$11^3 12^3 13^9$
8	5	2	11	$11^2 12^5\, 13^8$
7	7	1	11	$11\ 12^7\ 13^7$

(ii) $n = 15, \Delta$ is even and $\delta = \Delta - 2$: Here $n_{\Delta-1}$ is even.

n_Δ	$n_{\Delta-1}$	$n_{\Delta-2}$	$n_{\Delta-1}+2n_{\Delta-2}+2$	Impossible degree sequences
14	0	1	4	$(\Delta-2)\Delta^{14}$ for $\Delta \geq 6$
13	0	2	6	$(\Delta-2)^2\Delta^{13}$ for $\Delta \geq 8$
12	0	3	8	$(\Delta-2)^3\Delta^{12}$ for $\Delta \geq 10$
12	2	1	6	$(\Delta-2)(\Delta-1)^2\Delta^{12}$ for $\Delta \geq 8$
11	0	4	10	$(\Delta-2)^4\Delta^{11}$ for $\Delta \geq 12$
11	2	2	8	$(\Delta-2)^2(\Delta-1)^2\Delta^{11}$ or $\Delta \geq 10$
10	0	5	12	$12^5\ 14^{10}$
10	2	3	10	$(\Delta-2)^3(\Delta-1)^2\Delta^{10}$ for $\Delta \geq 12$
10	4	1	8	$(\Delta-2)(\Delta-1)^4\Delta^{10}$ for $\Delta \geq 10$
9	2	4	12	$12^4\ 13^2\ 14^9$
9	4	2	10	$(\Delta-2)^2(\Delta-1)^4\Delta^9$ for $\Delta \geq 12$
8	4	3	12	$12^3\ 13^4\ 14^8$
8	6	1	10	$(\Delta-2)(\Delta-1)^6\Delta^8$ for $\Delta \geq 12$
7	6	2	12	$12^2\ 13^6\ 14^7$
6	8	1	12	$12^1\ 13^8\ 14^6$

Corollary 4.3: If G is a Δ–critical graph with $\delta = \Delta - 2$, then $m \geq \frac{(\Delta-1)n+3}{2}$ if and only if $n_\Delta - n_{\Delta-2} \geq 3$.

Corollary 4.4: Let G be a Δ- critical graph of order 15 with $\delta = \Delta - 2$ and Δ be odd. Then
(i) If $n_\Delta = 13$, then $\Delta \leq 5$.
(ii) If $n_\Delta = 12$, then $\Delta \leq 7$. (iii) If $n_\Delta = 11$, then

$\Delta \le 9$. (iv) If $n_\Delta = 10$, then $\Delta \le 11$.

Corollary 4.5: Let G be a Δ- critical graph of order 15 with $\delta = \Delta - 2$ and Δ be even. Then (i) If $n_\Delta = 14$, then $\Delta \le 4$. (ii) If $n_\Delta = 13$, then $\Delta \le 6$. (iii) If $n_\Delta = 12$, then $\Delta \le 8$. (iv) If $n_\Delta = 11$, then $\Delta \le 10$. (v) If $n_\Delta = 10$, then $\Delta \le 12$.

Theorem 4.6: If G is a Δ- critical graph of odd order with $\delta = \Delta - 2$, $\Delta \ge 5$ and $n_{\Delta-2} = 1$, then $n_{\Delta-1} \ne 0$.

Proof: By theorem 4.1, $\Delta \le n_{\Delta-1} + 2n_{\Delta-2} + 2$.

Since $n_{\Delta-2} = 1, \Delta \le n_{\Delta-1} + 4$

$\Rightarrow n_{\Delta-1} \ge \Delta - 4 \ge 1 \Rightarrow n_{\Delta-1} \ne 0.$

Corollary 4.7: There is no Δ- critical graph G of odd order with $\Delta \ge 5$ whose degree sequence is $\pi(G) = (\Delta - 2)\Delta^{n_\Delta}$.

Corollary 4.8: If G is a 5-critical graph of odd order with $n_{\Delta-1} = 0$ and $\delta = 3$, then $n_{\Delta-2} > 1$.

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Dr. K. Kayathri/ Associate Professor/ Department of Mathematics
Thiagarajar College/ Madurai – 625 009/ kayathrikanagavel@gmail.com
J. Sakila Devi/Assistant Professor/
Department of Mathematics/Lady Doak College/ Madurai – 625 002/jsakiladevi@yahoo.com