# FUZZIFICATION OF FILTERS

**M. MUTHUKUMARI, A. NAGARAJAN, M. MURUGALINGAM**

**Abstract:** Filter is an important tool to characterize convergent functions and closure of a set in topological spaces. In this paper we try fuzzification of filters in topological spaces.Also we define alpha cut of fuzzy filters and study the properties.

**Keywords:** Fuzzy filter , α cut of Fuzzy filters.

**Introduction:** In a topological space filter is an important tool to study many properties. The closure of a set A can be characterized using convergent filters . The continuity of a function from one topological space to another can be characterized using convergent filters. In the year 1965 Lotfi A.Zadeh [2] introduced the concept of fuzzy sets and fuzzy logic.In the year 1968 C.L.Chang [1] introduced Fuzzy topological spaces. In the year 2014 We [3] introduced fuzzy sequences in a metric space. Also We [4] introduced fuzzy nets in topological spaces.

In this paper we try fuzzification of filters in topological spaces. The fuzzified filter in a topological space is called fuzzy filter. We study the properties of fuzzy filters.

**2. Fuzzy Filter:** First we recall the concept of filter in a topological space. Let X be a non empty set. F ⊂ P(X) is called a filter(crisp filter) on X if 1. ϕ ∉ F 2.F is closed under finite intersection. i.e., A, B ϵ F implies A∩B ϵ F. 3.B ϵ F and B ⊂ A implies A ϵ F. Now we define fuzzy filter

**Definition 2.1:** Let X be a non empty set. A fuzzy set f on P(X) is called a fuzzy filter [ f : P(X) → [0,1] ] if

1. f(ϕ) = 0
2. f(A∩B) ≥ min {f(A), f(B)}
3. B ⊂ A implies f(B) ≤ f(A)

**Example 2.2:** Let X = {1,2,3}. Define f : P(X) → [0,1] as f(ϕ) = 0, f({1}) = .1 f({2}) = 0, f({3}) = 0, f({1,2}) = .1, f({1,3}) = .5, f({2,3}) =0, f(X) = 1. f is a fuzzy filter on X.

**Theorem 2.3:** Let f be a fuzzy filter on X. Then f(A∩B) = min{f(A), f(B)}.

**Proof :** f is a fuzzy filter. Then f(A∩B) ≥ min {f(A), f(B)}……..(1). Since A∩B ⊂ A, we have f(A∩B) ≤ f(A). Also A∩B ⊂ B implies f(A∩B) ≤ f(B). Now f(A∩B) ≤ f(A) and f(A∩B) ≤ f(B) and hence f(A∩B) ≤ min {f(A), f(B)} …….(2) from (1) and (2) we have f(A∩B) = min {f(A), f(B)}

**Theorem 2.4:** Every crisp filter is a fuzzy filter.

**Proof :**

Let F be a crisp filter on X. Then F ⊂ P(X) where 1. ϕ ∉ F. 2. A, B ϵ F => A∩B ϵ F. 3. B ⊂ A, B ϵ F implies A ϵ F .Now define f : P(X) → [0,1] as f(A) = 1 if A ϵ F and 0 otherwise

Claim : f is a fuzzy filter.

1. F is crisp filter. Hence ϕ ∉ F. Therefore f(ϕ) = 0
2. Take A, B ϵ P(X)

Case : 1 Let A∩B ϵ F.

A∩B ⊂ A and A∩B ⊂ B. Hence A,B ϵ F.

Now f(A∩B) = 1, f(A) = 1, f(B) = 1.

Hence f(A∩B) = min { f(A), f(B) }

Case : 2 Let A∩B ∉ F. Hence f(A∩B) = 0.

If A ϵ F and B ϵ F then A∩B ϵ F. Therefore either A ∉ F or B ∉ F. Hence f(A) = 0 or f(B) = 0. Hence min {f(A), f(B)} = 0. Therefore f(A∩B) = min {f(A), f(B)}.

3. Let B ⊂ A

Case : 1. Let B ϵ F. Then f(B) = 1. Now B ⊂ A, B ϵ F implies A ϵ F and hence f(A) = 1. Therefore f(B) ≤ f(A)

Case : 2 Let B ∉ F. Then f(B) = 0. Now f(A) = 0 or 1. Hence f(B) ≤ f(A)

Hence f is a fuzzy filter. The given crisp filter F can be identified with the fuzzy filter f.

Every crisp filter can be considered as a fuzzy filter.

**Result 2.5:** Converse is not true.

A fuzzy filter need not be a crisp filter.

**Example 2.6:** Let X = {1,2,3}. Define f : P(X) → [0,1] as f(ϕ) = 0, f{1} = .1, f{2} =0, f{3} = 0, f{1,2} = .1 , f{1,3} = .2 , f{2,3} =0, f(X) = 1. Clearly f is a fuzzy filter. F takes values other than 0 and 1. Hence f is not a crisp filter.

**Theorem 2.7:** Intersection of two fuzzy filters on X is a fuzzy filter on X.

Proof :

Let f and g be two fuzzy filters on X. f : P(X) → [0,1] g : P(X) → [0,1]. Let h = f∩g. h is defined as h : P(X) → [0,1] where h(A) = min { f(A), g(A)}.

1. f and g are fuzzy filters. Hence f(ϕ) = 0, g(ϕ) = 0. Now h(ϕ) = min {f(ϕ), g(ϕ)} = min {0,0} = 0
2. let A, B ⊂ X

f is a fuzzy filter. Therefore f(A∩B) ≥ min {f(A), f(B)}. This implies f(A∩B) ≥ min {f(A), f(B), g(A), g(B)}. Hence f(A∩B) ≥ min {min {f(A), g(A)}, min {f(B), g(B)}. Therefore

f(A∩B) ≥ min {h(A), h(B)} ………..(1) Similarly

g(A∩B) ≥ min {h(A), h(B)} ………(2)

from (1) and (2) min {f(A∩B), g(A∩B) } ≥ min { h(A), h(B)}.Hence h(A∩B) ≥ min {h(A), h(B)}

3.Let B ⊂A. f and g are fuzzy filters.Therefore f(B) ≤ f(A) and g(B) ≤ g(A)Hence min{f(B), g(B)} ≤ min {f(A), g(A)}Hence f∩g (B) ≤ f∩g (A). hence h(B) ≤ h(A)From 1,2,3. h is a fuzzy filter.

**Result 2.8:** union of two fuzzy filters need not be a fuzzy filter.

**Example 2.9:** Let X = {1,2,3,4}

Let f and g be two fuzzy filters defined as follows

| | f | g | Let h = f∪g |
|---|---|---|---|
| ϕ | 0 | 0 | h(A) = max {f(A) |
| {1} | 5 | 0 | Take A = {1} B = {2} |
| {2} | 0 | .3 | A∩B = ϕ |
| {3} | 0 | 0 | h(A) = max {f(A) |
| {4} | 0 | 0 | = max {.5, .3} |
| {1,2} | .6 | .3 | h(A) = .5 |
| {1,3} | 5 | 0 | h(B) = max {f(B), g(B)} |
| {1,4} | 5 | 0 | = max {0, .3} |
| {2,3} | 0 | .3 | h(B) = .3 |
| {2,4} | 0 | .3 | min{h(A), h(B)} = min {.5, .3} |
| {3,4} | 0 | 0 | = .5 |
| {1,2,3} | .7 | .3 | h(A∩B) = max {f(A∩B), g(A∩B)} |
| {1,2,4} | .6 | .3 | = max {0,0} |
| {1,3,4} | 0.5 | 0 | h(A∩B)=0 |
| {2,3,4} | 0 | .3 | |
| X | 1 | 1 | |

h(A∩B) is not equal to min {h(A), h(B)}. Hence h is not a fuzzy filter. Therefore f∪g is not a fuzzy filter. Hence union of two fuzzy filters need not be a fuzzy filter.

**3. α cut of Fuzzy filter:** We recall the definition of α-cut of a fuzzy set. Let A be a fuzzy set on X. Let α ϵ [0,1]. Then α-cut of A denoted as αA is defined as αA = {x / A(x) ≥ α}. Now we see that α-cut of a fuzzy filter is a crisp filter.

**Theorem 3.1:** Let f be a fuzzy filter on a non empty set X. Let α ϵ Im F. Let α ≠ 0. Then α cut of f is a crisp filter on X.

Proof : X is a non empty set. f : P(X) → [0,1] is a fuzzy filter.

αf = {A ϵ P(X)/ F(A) ≥ α}.

We claim that αf is a crisp filter on X. Here α ≠ 0.

1. f(ϕ) = 0 implies f(ϕ) is not greater than or equal to α. Hence ϕ ∉ αf
2. Let A, B ϵ αf. Then f(A) ≥ α and f(B) ≥ α. This implies min{f(A), f(B)} ≥ α. Now f(A∩B) ≥ min {f(A), f(B)} ≥ α. Hence A∩B ϵ αf. Therefore A, B ϵ αf implies A∩B ϵ αf.
3. Let A ϵ αF and A⊂B. A ϵ αF implies F(A) ≥ α. A⊂B and F is a fuzzy filter. Therefore F(B) ≥ F(A). Hence F(B) ≥ α. Therefore B ϵ αF. Hence A ϵ αF and A⊂B implies B ϵ αF.

Therefore αf is a crisp filter on X.

**Result 3.2:** Converse is not true.

Let f : P(X) → [0,1] be a fuzzy set on P(X). αf is a crisp filter. Then f need not be a fuzzy filter.

**Example 3.3:** Let X = {a,b} Define f : P(X) → [0,1] as f(ϕ) = .2, f{a} = .6, f{b} = .1, f{a,b} = .7

Take α = .4. αf = { {a}, {a,b}}. αf is a crisp filter. But f(ϕ) = .2 ≠ 0. Hence f is not a fuzzy filter.

We give another example.

Let X = {a,b,c} Define f : P(X) → [0,1] as f(ϕ) = 0, f{a} = .4, f{b} = .1, f{c} = .1, f{a,b} = .1, f{a,c} = .1, f{b,c} = .5 f(X) = 1. Take α = .5

αf = {{b,c}, X}. αf is a crisp filter. Now {a} ⊂ {a,b} , f{a} = .4 and f{a,b} = .1. Hence f{a} is not less than or equal to f{a,b}. Hence f is not a fuzzy filter. Therefore α cut of a fuzzy filter is a crisp filter does not imply that f is a fuzzy filter.

Now we try find conditions on f such that converse is true.

**Theorem 3.4:** Let f : P(X) → [0,1] be a function where f(ϕ) = 0 and f is not zero map. If for every α ϵ Im f and α ≠0, α-cut of f is a crisp filter then f is a fuzzy filter.

Proof : f : P(X) → [0,1] is a function. f(ϕ) = 0. f is not zero map. α ≠ 0 and α ϵ Im f implies α-cut of f is a crisp filter. We claim that f is a fuzzy filter.

f : P(X) → [0,1]

1. f(ϕ) = 0 2. Let A,B ϵ P(X) . Let min{f(A), f(B)} = α. If α = 0, then f(A∩B) ≥ min {f(A), f(B)}

If α ≠ 0, consider αf. Nowα ≠ 0 and α ϵ Im f. Hence αf is a crisp filter. Since min{f(A), f(B)} = α, we have f(A) ≥ α and f(B) ≥ α. Hence A ϵ αf and B ϵ αf. αf is a crisp filter. Hence A∩B ϵ αf .Hence f(A∩B) ≥ α. Therefore f(A∩B) ≥ min {f(A), f(B)}. 3. Let A⊂B. Let f(A) = α. If α = 0 then f(B) ≥ f(A). If α ≠ 0 then α ≠ 0 and α ϵ Im f. Hence αf is a crisp filter. f(A) = α implies f(A) ≥ α. Hence A ϵ αf. Now A⊂B implies B ϵ αf. Hence f(B) ≥ α. Therefore f(B) ≥ f(A). Hence A ⊂ B implies f(B) ≥ f(A). Therefore f is a fuzzy filter on X.

**Conclusion :** When a crisp concept is fuzzified, it has many applications. We have fuzzified the concept of filters in a topological space. This can further be extended.

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M. Muthukumari / Research scholar , V.O.C. College, Tuticorin, Tamilnadu.
A. Nagarajan Associate Professor in Mathematics, V.O.C. College, Tuticorin, Tamilnadu /nagarajan.voc@gmail.com
M. Murugalingam / Associate Professor in Mathematics Thiruvalluvar College, Papanasam – 627425,Tamilnadu / mmmurugalingam@yahoo.com