ON GENERALISED CLOSED SETS IN NANO BITOPOLOGICAL SPACES

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Abstract: The paper introduces nano bitopological space by considering two equivalence relations on the same universe. Generalised closed sets in a nano bitopological space is defined. Some of the basic properties are discussed. The necessary and sufficient conditions for a set to be generalized closed set in a nano bitopological space is derived. The complement of the mentioned set is called the nano generalized open set. As an application a new separation axiom is introduced.

Keywords: Nano open set, Nano closed set, N(τ_R, τ_R)g-closed, N(τ_R, $\tau_{R'}$)g-open , N_{τ_R} Cl(A), N_{τ_R} Int(A)

Introduction: The notion of generalised closed sets introduced by Levine[5] plays a significant role in general topology. Kelly[1] introduced the concepts of bitopological spaces. The Rough set theory introduced by Pawlak[6,7] plays an important role in the fields of decision analysis, data analysis, pattern recognition etc. Nano topology in terms of approximations and boundary region of a subset of an universe using an equivalence relation on it was introduced by Lellis Thivagar et.al.[4] In this paper generalized closed set is introduced in a nano topology by considering two topologies on the same universe. Basic properties of the set are derived. The complement of the above mentioned set is called its respective open set. The necessary and sufficient condition for a set to be nano generalized closed and nano generalized open in a nano bitopological space is derived.

Preliminaries:

Definition 2.1[5] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Then U is divided into disjoint equivalence classes. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U,R) is said to be the approximation space. Let $X \subseteq$ U.

(i) The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by L_R(X).

L_R (X)=$\bigcup_{x\in U}$ $\{R\{x\}: R(x) \subseteq X\}$ where R(x) denotes the equivalence class determined by x$\in$ U

(ii) The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by U_R(X) .

U_R(X) = $\bigcup_{x\in U}\{R(x): R(x) \subseteq X\}$

(iii)The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as X with respect to R and it is denoted by B_R(X). B_R(X)= U_R(X) - L_R (X).

Definition 2.2[3] Let U be a non-empty, finite universe of objects and R be an equivalence relation on U. Let X$\subseteq$U. Let τ_R(X)=$\{\phi, U, L_R(X), U_R(X), B_R(X)\}$. Then τ_R(X) is a topology on U called as the nano topology with respect to X. Elements of the nano topology are known as the nano-open sets in U and (U, τ_R(X)) is called the nano topological space. [τ_R(X)]^c is called the dual nano topology of τ_R(X). Elements of [τ_R(X)]^c are called the nano closed sets.

Definition 2.3[4] A subset A of a nano topological space (U, τ_R(X) is called nano generalized closed if NCl(A) $\subseteq$ G whenever A$\subseteq$ G and G is nano open.

Definition 2.4[3] If (U, τ_R(X)) is a nano topological space with respect to X where X$\subseteq$ U and if A$\subseteq$ U, then the nano interior of A is defined as the union of all nano-open subsets of A and it is denoted by NInt(A). That is, NInt(A) is the largest nano-open subset of A. The nano closure of A is defined as the intersection of all nano closed sets ontaining A and it is denoted by NCl(A). That is NCl(A) is the smallest nano closed set containing A.

(i)NCl(NCl(A))=NCl(A)

(ii) NCl(A$\cup$ B)=Ncl(A) $\cup$ Ncl(B)

Generalized closed sets in a Nano Bitopological space:

Definition 3.1 Let U be a non-empty, finite universe of objects and R,R' are two equivalence relations on U. Let X $\subseteq$ U. Let τ_R(X)=$\{\phi, U, L_R(X), U_R(X), B_R(X)\}$ $\tau_{R'}$ (X)= $\{\phi, U, L_{R'}(X), U_{R'}(X), B_{R'}(X)\}$. Then the triplet (U, τ_R ,$\tau_{R'}$) is called the nano bitopological space. Since the same set X is used to define the two topologies τ_R and $\tau_{R'}$ will be used instead of τ_R(X) and $\tau_{R'}$ (X) respectively. The nano closure of a subset A of U with respect to the topologies τ_R(X) , $\tau_{R'}$ (X) will be denoted by $N_{\tau_R}Cl(A)$ and $N_{\tau_{R'}}Cl(A)$ respectively. The collection of all nano open and closed sets corresponding to the the topologies τ_R and $\tau_{R'}$ will be denoted by $N_{\tau_R}O(X), N_{\tau_{R'}}O(X), N_{\tau_R}C(X)$ and $N_{\tau_{R'}}C(X)$ $N_{\tau_{R'}}O(X)$ respectively. A nano closed subset of U with respect to the topology τ_R(X)will be called nano τ_R-closed.

Definition 3.2 In In a nano bitopological space (U, τ_R , $\tau_{R'}$) a subset of U is said to be nano (τ_R, $\tau_{R'}$) g-closed if $N_{\tau_{R'}}Cl(A) \subseteq$ U whenever A$\subseteq$ U where U $\in$ $N_{\tau_R}O(X)$.

The collection of all nano (τ_R, $\tau_{R'}$) g -closed sets on U will be denoted by N(τ_R , $\tau_{R'}$)GC(X).Similarly ($\tau_{R'}$, τ_R)GC(X) will be defined by replacing $N_{\tau_{R'}}Cl(A)$ by $N_{\tau_R}Cl(A)$

Example3.3 Let U= $\{a,b,c,d\}$ U/R= $\{\{a,b\},\{c\},\{d\}\}$ and X= $\{a,c\}$, $\tau_R(X)=\{\phi,U,\{a,b\},\{c\},\{a,b,c\}\}$,
U/R'= $\{\{a,d\},\{c\},\{b\}\}$,
$\tau_{R'}(X)=\{\phi,U,\{c\},\{a,d\},\{a,c,d\}\}$,
N(τ_R , $\tau_{R'}$) GC(X)=
$\{\phi,U,\{b\},\{d\},\{b,c\},\{a,d\},\{b,d\},\{c,d\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$
N($\tau_{R'}$, τ_R) GCX=
$\{\phi,U,\{b\},\{d\},\{a,b\},\{b,c\},\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\},\{b,c,d\}\}$

Remark 3.4 (i) If R=R' then a nano (τ_R , $\tau_{R'}$) g -closed set becomes a nano generalized closed set.
(ii) If τ_R is the indiscrete topology on U then N(τ_R , $\tau_{R'}$) GC(X) is the power set of U
(iii)If $\tau_{R'}$ is the indiscrete topology then
N($\tau_{R'}$, τ_R)GC(X)= $\{\phi$,U $\}$

Proposition 3.5 Every nano $\tau_{R'}$ -closed set is nano (τ_R , $\tau_{R'}$) g -closed .

Proof Let A be a nano $\tau_{R'}$ -closed set in U and G be any nano τ_R -open set containing A. Then $N_{\tau_{R'}}Cl(A)$ =A $\subseteq$ U. Hence A is nano (τ_R , $\tau_{R'}$) g-closed.

Remark3.6 The converse of the the proposition 3.5 is not true. In Example 3.3 the set $\{d\}$ is nano (τ_R , $\tau_{R'}$) g-closed but not $\tau_{R'}$ -closed.

Proposition 3.7 Union of two nano (τ_R , $\tau_{R'}$) g-closed sets is nano (τ_R , $\tau_{R'}$) g-closed.

Proof: Let A and B be two nano (τ_R , $\tau_{R'}$) g-closed sets in a nano bitopological space and G is a nano τ_R -open set containing A $\cup$ B. Since A and B are two nano (τ_R , $\tau_{R'}$) g-closed sets , $N_{\tau_{R'}}Cl(A)$ $\subseteq$ G, $N_{\tau_{R'}}Cl(B)$ $\subseteq$ G. $N_{\tau_{R'}}Cl(A\cup B)=N_{\tau_{R'}}Cl(A)\cup N_{\tau_{R'}}Cl(B)$ $\subseteq$ G. Hence A $\cup$ B is nano (τ_R , $\tau_{R'}$) g-closed.

Remark 3.8 Intersection of two nano (τ_R , $\tau_{R'}$) g-closed sets is not a nano (τ_R , $\tau_{R'}$) g -closed set. In Example 3.3 the sets $\{b,c\}$, $\{c,d\}$ are nano (τ_R , $\tau_{R'}$) g-closed but their intersection is not nano (τ_R , $\tau_{R'}$) g –closed.

Proposition 3.9 If $\tau_R\subseteq$ $\tau_{R'}$ in a nano bitopological space then N(τ_R , $\tau_{R'}$) GC(X) $\supseteq$ N($\tau_{R'}$, τ_R)GC(X).

Proof Let A be a nano (τ_R , $\tau_{R'}$) g –closed set and G be any nano τ_R -open set containing A. Since $\tau_R\subseteq$ $\tau_{R'}$ $N_{\tau_{R'}}Cl(A)\subseteq N_{\tau_R}Cl(A)$ and $N_{\tau_{R'}}Cl(A)\subseteq$ G. Hence A is nano (τ_R , $\tau_{R'}$) g –closed .

Proposition3.10 For each x $\in$ U , $\{x\}$ is either nano τ_R -closed or $\{x\}^c$ is nano (τ_R , $\tau_{R'}$) g –closed in a nano bitopological space.

Proof: Suppose $\{x\}$ is not nano τ_R -closed . Since $\{x\}^c$ is not nano τ_R -open, a nano τ_R -open set containing $\{x\}^c$ is U only. Then $N_{\tau_{R'}}Cl(\{x\}^c$ $\subseteq$ U. Hence $\{x\}^c$ is (τ_R , $\tau_{R'}$) g –closed.

Proposition 3.11 In a nano bitopological space $N_{\tau_{R'}}O(X)=N_{\tau_R}C(X)$ if and only if A is nano(τ_R , $\tau_{R'}$) g -closed and nano ($\tau_{R'}$, τ_R) g-closed for any subset A of U.

Proof: Necessity. Let A be a subset of U and G be a nano $\tau_{R'}$ -open set containing A. By hypothesis $N_{\tau_R}Cl(A)\subseteq$ G and hence A is nano ($\tau_{R'}$, τ_R) g -closed. Since $N_{\tau_{R'}}O(X)=N_{\tau_R}C(X)$, A is nano (τ_R , $\tau_{R'}$)g-closed. Sufficiency.Suppose A is nano $\tau_{R'}$ -open. Since A is nano ($\tau_{R'}$, τ_R)g-closed , $N_{\tau_R}Cl(A)\subseteq$ A. Hence A is nano τ_R -closed. Conversely let B be nano τ_R -closed. Since B^c nano(τ_R , $\tau_{R'}$) g-closed and nano τ_R -open, $N_{\tau_{R'}}Cl(B^c)\subseteq$ B^c. Hence B^c is nano $\tau_{R'}$ -closed and hence B is nano $\tau_{R'}$ -open. $N_{\tau_{R'}}O(X)=N_{\tau_R}C(X)$.

Proposition 3.12 (i)In a nano bitopological space if a subset A of U is nano (τ_R , $\tau_{R'}$) g–closed then $N_{\tau_{R'}}Cl(A)$ -A contains no non empty nano τ_R -closed set.
(ii) A is nano nano (τ_R , $\tau_{R'}$) g -closed if and only if $N_{\tau_R}Cl(\{x\})\cap A\neq\phi$ for each x $\in$ $N_{\tau_{R'}}Cl(A)$

Proof: (i) Let F be a nano τ_R -closed set contained in $N_{\tau_{R'}}Cl(A)$ -A.Since A is nano (τ_R , $\tau_{R'}$) g –closed, $N_{\tau_{R'}}cl(A)\subseteq F^c$.
Therefore F $\subseteq$ $N_{\tau_{R'}}Cl(A)$ $\cap$ ($N_{\tau_{R'}}Cl(A)$)^c. Hence F= ϕ .
(ii) Necessity Suppose $N_{\tau_R}Cl(\{x\})\cap A=\phi$ for some x $\in$ $N_{\tau_{R'}}Cl(A)$ .Then A $\subseteq$ $[N_{\tau_R}Cl(\{x\})]^c$. Since $[N_{\tau_R}Cl(\{x\})]^c$ is nano τ_R -open then $N_{\tau_{R'}}Cl(A)\subseteq[N_{\tau_R}Cl(\{x\})]^c$. This implies that x $\notin$ $N_{\tau_{R'}}Cl(A)$. This contradicts the assumption.Hence $N_{\tau_R}Cl(\{x\})\cap A\neq\phi$ for each x $\in$ $N_{\tau_{R'}}Cl(A)$ .Conversely let G be any nano τ_R -open set such that A $\subseteq$ G. Suppose x $\in$ $N_{\tau_{R'}}Cl(A)$ then $N_{\tau_R}Cl(\{x\})\cap A\neq\phi$ by assumption. There exist a point y

such that y∈ $N_{\tau_{R'}}Cl(A)$ and y∈ A ⊆ G. This implies that G ∩ $\{x\} \neq \phi$. i.e x∈G. Thus $N_{\tau_{R'}}Cl(A) \subseteq$ G. Hence A is nano $(\tau_R, \tau_{R'})$ g –closed.

Remark 3.13 The converse of the first part of the proposition 3.12 is not true.

Example 3.14 Let U= $\{a,b,c,d\}$ U/R= $\{\{a,b\},\{c\},\{d\}\}$ and X= $\{a,c\}$, $\tau_R(X) = \{\phi, U, \{a,b\},\{c\},\{a,b,c\}\}$, U/R'= $\{\{a,c\},\{d\},\{b\}\}$, $\tau_{R'}(X) = \{\phi, U, \{a,c\}\}$, N($\tau_R, \tau_{R'}$) GC(X)= $\{\phi, U, \{d\},\{a,d\},\{b,d\},\{c,d\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$ $\tau_{R'}$ Cl($\{b,c\}$)- $\{b,c\}$ = $\{a,d\}$ which does not contain any non empty nano $\tau_{R'}$ -closed set but $\{b,c\}$ is not nano $(\tau_R, \tau_{R'})$ g –closed.

Proposition 3.15 If A is nano $(\tau_R, \tau_{R'})$ g –closed and A ⊆ B ⊆ $N_{\tau_{R'}}Cl(A)$ then B is nano $(\tau_R, \tau_{R'})$ g –closed.

Proof: Let B ⊆ G where G is nano τ_R -open. Then A ⊆ G. Since A is nano $(\tau_R, \tau_{R'})$ g –closed $N_{\tau_{R'}}Cl(A) \subseteq$ G, $N_{\tau_{R'}}Cl(B) \subseteq N_{\tau_{R'}}Cl(A) \subseteq$ G. $N_{\tau_{R'}}Cl(B) \subseteq$ G. Hence B is nano $(\tau_R, \tau_{R'})$ g-closed.

Proposition 3.16 If A is nano $(\tau_R, \tau_{R'})$ g-closed in a nano bitopological space then A is nano $\tau_{R'}$ -closed if and only if $N_{\tau_{R'}}Cl(A)$ -A is nano τ_R -closed.

Proof: : If A is nano $\tau_{R'}$ -closed then $N_{\tau_{R'}}Cl(A)$ -A= ϕ (By the Proposition 3.12(i)) Hence $N_{\tau_{R'}}Cl(A)$ -A is nano τ_R -closed. Sufficiency. If $N_{\tau_{R'}}Cl(A)$ -A is nano τ_R -closed then by the Proposition 3.12 (i) $N_{\tau_{R'}}Cl(A)$ -A= ϕ . Hence A is nano τ_R .-closed.

Theorem 3.17 In a nano bitopological space $N_{\tau_R}O(X) \subseteq N_{\tau_{R'}}C(X)$ if and only if every subset of U is nano $(\tau_R, \tau_{R'})$ g-closed.

Proof: Suppose $N_{\tau_R}O(X) \subseteq N_{\tau_{R'}}C(X)$ .Let A be a subset of G where G isnano τ_R -open.Then $N_{\tau_{R'}}Cl(A) \subseteq N_{\tau_{R'}}Cl(G)$ =G. Hence A is nano $(\tau_R, \tau_{R'})$ g-closed . Conversely suppose that every subset of U is nano $(\tau_R, \tau_{R'})$ g-closed. Let G be a nano τ_R -open set. Since G is nano $(\tau_R, \tau_{R'})$ g-closed $N_{\tau_{R'}}Cl(G) \subseteq$ G. Hence G is nano $\tau_{R'}$ -closed.Thus $N_{\tau_R}O(X) \subseteq N_{\tau_{R'}}C(X)$.

Proposition 3.18 If A ⊆ B ⊆ U and A is nano $(\tau_R, \tau_{R'})$ g-closed then A is nano $(\tau_R, \tau_{R'})$ g-closed relative to B.

Proof: Let G be any nano τ_R -open set in B such that A ⊆ G. Then G=H ∩ B for some nano τ_R -open set H in U. Thus A ⊆ H ∩ B. Since A is nano $(\tau_R, \tau_{R'})$ g- closed in U, $N_{\tau_{R'}}Cl(A) \subseteq H$. Therefore B ∩ $N_{\tau_{R'}}Cl(A) \subseteq$ H ∩ B. That is A is nano $(\tau_R, \tau_{R'})$ g-closed relative to B.

Definition 3.19 A subset A of U in a nano bitopological space is said to be nano $(\tau_R, \tau_{R'})$ g-open if its complement is nano $(\tau_R, \tau_{R'})$ g -closed in U.

Theorem 3.20 A subset A of a nano bitopological space is nano $(\tau_R, \tau_{R'})$ g-open if and only if F ⊆ $N_{\tau_{R'}}Int(A)$ whenever F is nano τ_R -closed and F ⊆ A.

Proof: Necessity.Let A be a nano $(\tau_R, \tau_{R'})$ g-open set in U and F be a nano τ_R -closed set such that F ⊆ A. Then $A^c \subseteq F^c$ where F^c is nano τ_R -open and A^c is nano (τ_R , $\tau_{R'}$) g–closed implies $N_{\tau_{R'}}Cl(A^c) \subseteq F^c$, $(N_{\tau_{R'}}Int(A))^c \subseteq F^c$ .Hence F ⊆ $N_{\tau_{R'}}Int(A)$ .Sufficiency.Let F ⊆ $N_{\tau_{R'}}Int(A)$ where F is nano τ_R -closed and F ⊆ A. Then $A^c \subseteq F^c$=G and G is nano τ_R -open. Then $G^c \subseteq$ A implies $G^c \subseteq N_{\tau_{R'}}Int(A)$ or $N_{\tau_{R'}}Cl(A^c) = [N_{\tau_{R'}}Int(A)]^c \subseteq$ G. Thus A^c is nano $(\tau_R, \tau_{R'})$ g-closed or A is nano ($\tau_R, \tau_{R'}$)g-open.

Application:

Definition 4.1 A nano bitopological space is said to be a τ_R $\tau_{R'}$ T_0-space if for each pair of distinct points x,y of U there exists a nano $(\tau_R, \tau_{R'})$ g-open set G containing x but not y or a nano $(\tau_R, \tau_{R'})$ g-open set H containing y but not x.

Example4.2 Let U= $\{a,b,c,d\}$ U/R= $\{\{a,b\},\{c,d\}\}$ and X= $\{a,c,d\}$, $\tau_R(X) = \{\phi, U, \{a,b\},\{c,d\}\}$, U/R'= $\{\{a,c\},\{d\},\{b\}\}$, $\tau_{R'}(X) = \{\phi, U, \{a,c,d\}\}$, N($\tau_R, \tau_{R'}$) GC(X)= $\{\phi, U, \{b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$ N($\tau_{R'}, \tau_R$) GCX= $\{\phi, U, \{a\},\{b\},\{c\},\{d\},\{b,d\},\{b,c\},\{a,d\},\{a,c\},\{a,c,d\}\}$ The Nano bitopological space (U, $\tau_R, \tau_{R'}$) is a nano τ_R $\tau_{R'}$ T_0-space.

N($\tau_R, \tau_{R'}$) GO(X) is the collection of all nano $(\tau_R, \tau_{R'})$ g-open sets in U.

Theorem 4.3 For a nano bitopological space (U, $\tau_R, \tau_{R'}$) the following are equivalent. .

(i) (U, $\tau_R, \tau_{R'}$) is a nano τ_R $\tau_{R'}$ T_0-space .

(ii) For each x∈ U ,
$\{x\} = \bigcap\{F \in N(\tau_R, \tau_{R'})GC(X) \cup N(\tau_R, \tau_{R'})GO(X) : x \in F\}$

proof: (i) $\Longrightarrow$ (ii) By definition $\{x\} \subseteq \bigcap\{F \in N(\tau_R, \tau_{R'})GC(X) \cup N(\tau_R, \tau_{R'})GO(X) : x \in F\}$.
To prove the other inequality we need to prove that $\bigcap\{F \in N(\tau_R, \tau_{R'})GC(X) \cup N(\tau_R, \tau_{R'})GO(X) : x \in F\}$ contains only x. If y∈ U, y ≠ x. Two cases arise (i) x∈ G, y∉ G.

(ii) y∈ H, x∉ H where G,H are nano ($\tau_R, \tau_{R'}$)-open sets. By case

(i) $y \notin \bigcap\{F \in N(\tau_R, \tau_{R'})GC(X) \cup N(\tau_R, \tau_{R'})GO(X) : x \in F\}$.

By case

(ii)x∈ H^c

$y \notin \bigcap\{F \in N(\tau_R, \tau_{R'})GC(X) \cup N(\tau_R, \tau_{R'})GO(X) : x \in F\}$.

Hence $\bigcap\{F \in N(\tau_R, \tau_{R'})GC(X) \cup N(\tau_R, \tau_{R'})GO(X) : x \in F\}$ contains only x.

(ii) $\Longrightarrow$ (i) Let x,y be any two distinct points of (U, $\tau_R, \tau_{R'}$). Then (ii) implies that there exists a nano ($\tau_R, \tau_{R'}$)g-open set G which contains x then it does not contain y. Hence (U, $\tau_R, \tau_{R'}$) is τ_R $\tau_{R'}$ T_0-space.

Conclusion: The nano generalized closed in a nano bitopological space coincides with the nano generalized closed set if the two topologies defined on the universe are same. In a particular case the collection of all nano generalized closed set in nano bitopological space is the power set of U. This concept can be extended to derive a new decomposition of continuity in a nano bitopological space. As an application a new separation axiom is introduced.

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