CHARACTERIZATION OF ZERO DIVISOR GRAPHS OF CONNECTED RIGNS

A. SWETHA, G.SHOBHALATHA

Abstract: Anderson and Livingston studied the properties of the zero divisor graph of a commutative ring. In this paper, we present the Characterization of the zero divisor graph of a connected ring. A connected ring (R, +, . , o) is a ring (R, +, .) with the connected operation o, that is, x o $y = x$ a y for any x, a, y in R. We prove that if R is a commutative connected ring. Then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex if and only if either $R \cong Z_2 \times A$; where R is an integral domain, or $Z(R)$ is an annihilator ideal (and hence is prime). Also if R be a finite commutative connected ring with $|\Gamma(R)|\geq 4$. Then $\Gamma(R)$ is a star graph if and only if $R \cong Z_2 \times F$, where F is a finite field. In particulars if $\Gamma(R)$ is a star graph, then $\Gamma(R) = p^n$ for some prime p and integer $n \geq 0$. Conversely, each star graph of order p^n can be realized as $\Gamma(R)$.

Keywords: connected ring, integral domain, zero divisor graph ,annihilator ideal, Noetherian ring , complete graph , local ring, Star graph.

Introduction: Anderson and Livingston [1] studied the properties of the zero divisor graph of a commutative ring. In this paper we present the Characterization of the zero divisor graph of a connected ring. Throughout this paper R denotes a commutative connected ring with identity element 1 and Z (R) be its set of zero-devisors . Γ (R) denotes a graph associated to R such that the vertices of Γ (R) of the elements of Z(R)*' where Z(R)* is the set of non-zero zero-divisors of R. The vertices x and y are adjacent if and only if x o $y = x$ a y=0 for any a in R. This Γ(R) is called a zero-divisor graph of R. Γ (R) is an empty graph if and only if R is an integral domain. A commutative ring R is called an integral domain if $x.y = 0$ for all $x,y \in R$ implies $x = 0$ or $y = 0$. If one vertex is singleton in a complete bipartite graph then it is called a star graph. It is denoted by $K_{1,n}$.

Ring theoretic preliminaries and Graph theoretic preliminaries

Connected Ring : Let $(R, +, \cdot , \circ)$ is said to be a connected ring if it satisfies the following:

(i) $(R, +)$ is an abelian group

(ii) $(R, \cdot)$ is a semi ring

(iii) The operation is distributive over + that is $x \cdot (y + z) = x \cdot y + x \cdot z$, $(x + y) \cdot z = x \cdot z + y . z$

(iv) The operation $\circ$ is connected, that is $x \circ y = xay$ for all x, y, a in R.

Integral domain:A commutative ring R is called an integral domain if $x.y = 0$ for all $x,y\in R$ implies $x = 0$ or $y = 0$.

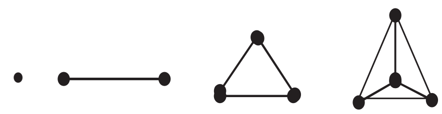
Annihilator: Let R be a commutative ring and S be a non-empty subset of R. Let $Ann (S) = \{x \in R/Sx = xS = 0\}$.
Then $Ann (S)$ is called annihilator of S. In fact $Ann(S)$ is an ideal in R .

Local ring*:* A ring R with unity is called a local ring if it has a unique maximal right ideal.

Noetherian ring: A ring R with identity satisfying acc on ideals is called a Noetherian ring.

Complete Graph : A graph G in which every vertex is adjacent to every other vertex in G is called a complete graph. Complete graph is represented as K_n. where n is the number of vertices in K_n.

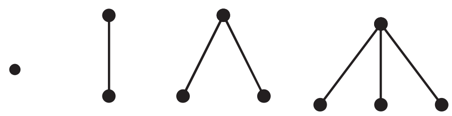
Ex:



K_1 K_2 K_3 K_4

Star graph : If one vertex is singleton in a complete bipartite graph then it is called a star graph. It is denoted by K1,n.

Ex:



$K_{1,0}$ $K_{1,1}$ $K_{1,2}$ $K_{1,3}$

Zero – divisor graph: A zero-divisor graph □(R) associated to R is the graph whose vertices are the elements of Z(R)*where. $Z(R)^* = Z(R) - \{0\}$, the set of non-zero zero-divisor of R and the vertices x and y are adjacent if and only if $xy = 0$.

3. Main theorems: We next determine when $\Gamma(R)$ has a vertex adjacent to every other vertex.

Theorem 3.1: Let R be a commutative connected ring. Then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex if and only if either $R \cong Z_2 \times A$; where R is an integral domain, or $Z(R)$ is an annihilator ideal (and hence is prime).

Proof: Let R be a commutative connected ring. Assume that $Z(R)$ is not an annihilator ideal.

Let $0 \neq x \in Z(R)$ be a adjacent to every other vertex.

Then $x \notin ann\ (x) = I$, other wise $Z(R) - I$ would be an annihilator ideal.

So I is maximal among annihilator ideals, and hence is prime [2].

If $xax \neq x$ then $(xaxax) = (xax)ax = 0$, and hence $x \in I$, a contradiction. So $xax = x$ and $R = Rx \oplus R(1 - x)$.
Hence we may assume that $R = R_1 \times R_2$ with (1,0) adjacent to every other vertex.
Now for any $1 \neq c \in R_1$ $(c,0)$ is a zero – divisor.
For $(s,t) \in R_1$ x R_2, $(c,0)0\ (s,t) = (c,0)\ (0,0)\ (s,t) = (0,0)\ (s,t)$.
Since (1,0) is adjacent to every other vertex it follows that $(1,0)\ (s,t) = (0,0)$.
i.e., $(s,0) = (0,0) \Rightarrow s = 0$. Then $(c,0) = (c,0)\ (0,0)\ (1,0) = (0,0)$,
a contradiction unless $c = 0$. Hence $R_1 \cong Z_2$.
Suppose R_2 is not an integral domain. Then there is some $b \in Z(R_2)$ with $b \neq 0$.
Now (1,1) is a zero-divisor of R. But $(1,b)$ is not adjacent to (1,0) since $(1,b) \circ (1,0) = (1,b)\ (0,0)\ (1,0) = (1,0) \neq (0,0)$, a contradiction to our assumption. Thus R_2 must be an integral domain. Therefore $R \cong Z_2$ x A, where A is an integral domain.
If $Z(R)$ is not an annihilator ideal we have shown that $R \cong Z_2 \times A$ where A is an integral domain.
Suppose $Z(R)$ is an annihilator ideal. Then it is certainly maximal among annihilator ideal and hence is prime.
Conversely, suppose that $R \cong Z_2 \times X$ where A is an integral domain. Then $(\overline{1},0)$ is adjacent to every other vertex.
If $Z(R) = Ann\ x$ for some $x \in R$ with $x \neq 0$, then x is adjacent to every other vertex.
The proof of the following corollaries can be found in [2]

Corollary 3.1: Let R be a commutative Noetherian ring. Then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex if and only if either $R \cong Z_2 \times A$, where A is a field or {0} is primary ideal of R. i.e., $Z(R) = \text{nill}\ (R)$.

Corollary 3.2: Let R be a commutative connected ring. Then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex if and only if either $R \cong Z_2$ x F or R is local where F is a finite field. Moreover, for some prime p and integer $n \geq 1$ $|\Gamma(R)| = |F| = p^n$ if $R \cong Z_2$ x F, and $|\Gamma(R)| = p^n - 1$ if R is local.

We next discuss when $\Gamma(R)$ is a complete graph. By the definition of $\Gamma(R)$. $\Gamma(R)$ is complete if and only if $xay = 0$ for all distinct $x,y \in Z(R)$. We must also have $xax = 0$ for all $x \in Z(R)$ where $\Gamma(R)$ is complete except for the case when $R \cong Z_2 \times Z_2$. So except for this case, nilpotent elements are detected by complete graphs.

Theorem 3.2: Let R be a commutative connected ring. Then $\Gamma(R)$ is complete if and only if $R \cong Z_2 \times Z_2$ or $xay = 0$ for all $x,y \in Z(R)$.

Proof: Let R be a commutative connected ring suppose and $\Gamma(R)$ is a complete graph i.e., any two distinct vertices are adjacent. Then there is an $x \in Z(R)$ with $xax \neq 0$.
We have to show that $xax = x$
If not, then $xaxax = (xax)ax$
$= 0$
Hence $xax(x + x \circ x) = (xaxax) + (xaxaxax)$
$= (xaxax) + (xaxax).x$
$= 0 + 0.x$
$= 0$
$xax(x+xax) = 0$
with $xax \neq 0$. So, $x + xax \in Z(R)$.
If $x = xax = x$ then, $xax = 0$. A contradiction,
$x + xax \neq x$, and so, xax.
$\Rightarrow (xax) + (xaxax)$
$\Rightarrow x(x+xax)$
$\Rightarrow 0$.
Since $\Gamma(R)$ is complete, this is a contradiction.
Hence $xax = x$.
Then as in the proof of theorem 3.1 we have $R \cong Z_2 \times A$.
But necessarily $A \cong Z_2$.
Therefore $R \cong Z_2 \times Z_2$.
Conversely, suppose that $R \cong Z_2$ x Z_2 or $xay = 0$ for all $xay \in Z(R)$.
By theorem 3.1 in $\Gamma(R)$ every vertex is adjacent to other vertex.
Hence $\Gamma(R)$ is complete.

Corollary 3.3: Let 'R' be a commutative connected ring. For $x,y \in Z(R)$, define $x \sim y$ is $xay= 0$ or $x = y$, and define $x \sim * y$ if $xay=0$.

(a) The relation ~ is transitive (equivalently, an equivalence relation if and only if $\Gamma(R)$ is complete.
(b) Then relation ~* is transitive (equivalently, an equivalence relation) if and only if $\Gamma(R)$ is complete and $R \cong Z_2$ x Z_2.

Proof: Let R be a commutative connected ring. For $x,y \in Z(R)$, define $x \sim y$ if xay or $x= y$.

(a) Suppose the relation ~ is transitive. i.e., for $x,t,y \in Z(R)$.
$x \neq y$, if $x \sim t$, $t \sim y$ then $x \sim y$, i.e., $xay=0$.
By theorem 3.1 in $\Gamma(R)$ every vertex is adjacent to other vertex.
Hence $\Gamma(R)$ is complete.
Suppose $\Gamma(R)$ is complete. Then every two vertices is $\Gamma(R)$ are adjacent. This implies $xay = 0$ for $x,y \in Z(R)$.
So if $x \sim t$ and $t \sim y$ then $x \sim y$, since $xay=0$. That is the relation ~ is transitive.
(b) Suppose the relation ~* transitive. That is $x{\sim}t^*$ and $t{\sim}^*y$ then $x{\sim}^*y$. T his implies $xay = 0$ and hence $\Gamma(R)$ iscomplete.
Suppose $\Gamma(R)$ is complete.Then for $x,y \in Z(R)$, $xay = 0$.Let $x,t,y \in Z(R)$ such that $x \sim * t$ and $t \sim * y$.
Since $xay = 0$ we have $t \sim * y$.
That is the relation ~* is transitive.
Then as in the proof of theorem 3.1 we have $R \cong Z_2 \times A$.But necessarily $A \cong Z_2$. Therefore $R \cong Z_2 \times Z_2$.

Theorem 3.3: Let R be a finite commutative connected ring. If $\Gamma(R)$ is complete, then either $R \cong Z_2 \times Z_2$ or R is local with char $R = p$ or p^2 $|\Gamma(R)| = p^n - 1$, where p is prime and $n \geq 1$.

Proof: let F be a field, we have seen in theorem 3.2 that $\Gamma(R)$ is complete if and only if $R \cong Z_2 \times Z_2$. So the graph $\Gamma(Z_2 \times F)$ is not complete unless $F=Z_2$.
So assume that $R \ncong Z_2 \times Z_2$. We show that R is local with char $R = p$ or p^2. We know R must be local with

maximal ideal M.
Hence char $R = p^m$ for some p and integer $m \geq 1$.
If $m \geq 3$, then R would have non – adjacent zero – divisors a contradiction.
Hence char $R = p$ or p^2.
Since M is a p – group it follows that
$|\Gamma(R)| = p^n - 1$,
for some prime p and integer $n \geq 1$.

Theorem 3.4: Let R be a finite commutative connected ring. If $\Gamma(R)$ has exactly one vertex adjacent to every other vertex and no other adjacent vertices, then either $R \cong Z_2 \times F$, where F is a finite field with $|F| \geq 3$, or R is local with maximal ideal M satisfying
$R/M \cong Z_2$, $MaMaM = 0$ and $|MaM| \leq 2$. Thus $\Gamma(R)$ is either p^n or $2^n - 1$ for some prime p and integer $n \geq 1$.

Proof: Let R be a finite commutative connected ring. Let $\Gamma(R)$ has exactly one vertex adjacent to every vertex. If $R \ncong Z_2$ x F, then by corollary 3.3, R is local with maximal idea M.
Thus $M = Ann\ X$ for a unique $x \in M$.
Then $M^k = 0$ where k is the least positive integer.
Then for any $y \neq 0$, $y \in M^{k-1}$
We have $M = Ann\ y$. So $M^{k-1} = \{0.x\}$, and
Thus $|M^{k-1}/M^k| = 2$.
This yields $R/M \cong Z_2$.
If $k \geq 4$ then $|M^{k-2}| \geq 4$.
So for any $a \neq b$ such that $a,b \in M^{k-2} - M^{k-1}$, we have $axb \in M^{2k-4} \subset M^k$.
Since $M^k = 0$ this imply that $axb = 0$, a contradiction.
Hence $MaMaM = 0$ and $|MaM| \leq 2$.
If $R \cong Z_2 \times F$ then $|\Gamma(R)| = |F| = p^n$ by corollary 3.3 since R is local we have $|\Gamma(R)| = 2^n - 1$ by corollary 3.3.
Thus $|\Gamma(R)| = P^n$ or $2^n - 1$.

Remark: We know $\Gamma(Z_2 \times Z_7)$ is a star graph of order $|Z_7|$ we use Theorem 3.4 to show that except for zero-divisor graphs of small order, this is the only way a star graph arises as $\Gamma(R)$.
The following theorem establishes how star graph arise as $\Gamma(R)$.

Theorem 3.5 : Let R be a finite commutative connected ring with $|\Gamma(R)| \geq 4$. Then $\Gamma(R)$ is a star graph if and only if $R \cong Z_2 \times F$, where F is a finite field. In particulars if $\Gamma(R)$ is a star graph, then $\Gamma(R) = p^n$ for some prime p and integer $n \geq 0$. Conversely, each star graph of order p^n can be realized as $\Gamma(R)$.

Proof: Let R be a finite commutative connected ring with $|\Gamma(R)| \geq 4$. Suppose $\Gamma(R)$ is a star graph.
It possible let $R \cong Z_2 \times F$.
Then R is local.
We assume that (R,M) is local with
$|M| = 2^k$ for some $k \geq 3$
since M is a 2 – group and $|\Gamma(R)| \geq 4$ and $MaM = 2$ let $M = Ann\ x$. Choose distinct $a,b,c,d \in M^* - \{x\}$.
Now $axb = axc = axd = y$ and there are no zero-divisor relations, since $Mam = \{0,x\}$.
Hence $ax(b - c) = ax\ (b - d) = 0$.
Then we have $Anna = \{0,x\}$ since $Anna$ has even order, and so $a \notin Ann\ a$.
Thus $b - c = b - d = X$.
Hence $c = d$.
This is a contradiction to c,d are distinct.
Therefore $R \cong Z_2 \times F$.
Suppose $\Gamma(R)$ is a star graph. Then these is one vertex which adjacent to every other vertex.
Then by Lemma $|\Gamma(R)| = p^h$, since $R \cong Z_2 \times F$. Conversely, let G be a star graph of order p^h.
Let F be a finite field of order p^h.
Let $R \cong Z_2 \times F$.
Then $\Gamma(R)$ is a star graph.
That is, every star graph is realized as $\Gamma(R)$.

Acknowledgement: I am very much thankful to UGC-RGNF for providing me good financial support. I am very much thankful to my Research guide Prof,K.Suvarna mam, Prof,G.Shobha latha mam and Department of mathematics .And finally I thank to my research colleagues for there best support through out my research work.

References:

1. Anderson D.F and Livingston P.S, "The zero-divisor graph of a commutative ring", J . Algebra 217 (1999), 434-447.
2. Kaplansky I, "Commutative Rings", Rev. ed., Univ. of Chicago press, Chicago, 1974.
3. Anderson D.F, Frazier A, Lauve A, and Livingston P.S, "The zero-divisor graph of a commutative ring 11". Lecture Notes in pure and Appl. Math. 220 (2001), 61-72.
4. Atiyah M.F and Mac Donald I.G, "Introduction to Commutative Algebra". Addison – Wesley, Reading, M.A., 1969.
5. Ganesan N, "Properties of rings with a finite number of zero-divisors II", Math. Ann. 161 (1965), 241-246.
6. Smith, N.O, "Planar zero-divisor Graph", International Journal of Commutative Rings", 2002, 2(4), 177-188.

***

A. Swetha/Research Scholar, Department of Mathematics S.K.University, Anantapur (A.P.),India
G.Shobhalatha /Professor/ Department of Mathematics/ S.K.University/ Anantapur(A.P.)/ India