SOME SEPARATION PROPERTIES OF ČECH SOFT CLOSURE SPACES

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Abstract: In this paper we study some separation axioms such as soft T_0-space, soft T_1-space in Čech soft closure spaces.

Keywords: Čech soft closure spaces, soft T_0-space, soft T_1-space.

Introduction: In 1999, D. Molodtsov[4] introduced the concept of soft sets in order to solve complicated problems in some sciences such as economics, engineering etc. Later, he applied this theory to several directions. The soft set theory has been applied to many different fields. In fact the soft set theory has a rich potentialfor application in solving practical problems in economics, engineering environment, Social Science, Medical Scince and business management. In 2011, Shabir and Naz[5] introduced and studied the concepts of soft topological spaces. Recent years, the theory of soft topological spaces is investigated by various new researchers. In [2], we introduced Čech Soft Closure Spaces over the soft sets on a non-empty set X and we exhibit some results related to these concepts. We defined some separation axioms in Čech Soft Closure Spaces. The basic definitions and notations we follow [4, 5].

Čech Soft Closure Spaces:

Definition 2.1. An operator F : S(X, E) → S(X, E) is defined on the set of all soft sets S(X, E) of a set X is called *Čech Soft Closure operator* on X if the following three axioms are satisfied:

(C1) $\tilde{c}(\phi) = \phi$,

(C2) $F_A \tilde{\subseteq} \tilde{c}(F_A)$, for all soft sets F_A over X,

(C3) $\tilde{c}(F_A \tilde{\cup} F_B) = \tilde{c}(F_A) \tilde{\cup} \tilde{c}(F_B)$, for all soft sets F_A, F_B over X.

Then $\tilde{c}$, together with the underlying set X, is called a *Čech Soft Closure Space* and is denoted by $(X, \tilde{c}, E)$.

If $\tilde{c}$ also satisfies:

(C4) $\tilde{c}(\tilde{c}(F_A)) = \tilde{c}(F_A)$ for all F_A over X, then $(X, \tilde{c}, E)$ is a *soft topological space* .

Definition 2.2. Let $\tilde{c}$ and $\tilde{d}$ be two Čech soft closure operators on a set X. $\tilde{c}$ is said to be *coarser* than $\tilde{d}$, or equivalently $\tilde{d}$ is *finer* than $\tilde{c}$, if $\tilde{d}(F_E) \tilde{\subseteq} \tilde{c}(F_E)$, for each soft set F_E over X.

Definition 2.3. Let $(X, \tilde{c}, E)$ be Čech soft closure space . A soft subset F_E over X is called *soft closed* provided $F_E = \tilde{c}(F_E)$. A soft subset F_E over X is called *soft open* provided its soft complement $\tilde{X}$ – F_E is soft closed.

Definition 2.4. Let $(X, \tilde{c}, E)$ be Čech soft closure space. G_E be a soft set over X and $x \in X$ .Then x is said to be a *soft interior point* of G_E if there exists a soft open set F_E such that $x \tilde{\in} F_E \tilde{\subseteq} G_E$.

Definition 2.5 . Let $(X, \tilde{c}, E)$ be Čech soft closure space. G_E be a soft set over X and $x \in X$ .Then G_E is said to be a *soft neighbourhood of x* if there exists a soft open set F_E such that $x \tilde{\in} F_E \tilde{\subseteq} G_E$.

Remark 2.6. For each Čech soft closure space, there exists an underlying soft topological space that can be defined in a natural way. Some familiarity with the rudiments of topology on the part of the reader is assumed. If $(X, \tilde{c}, E)$ is a Čech soft closure space, we denote the associated soft topology on X by $\tilde{\tau}(\tilde{c})$. That is $\tilde{\tau}(\tilde{c})$ = { F_E' : $\tilde{c}(F_E) = F_E\}$ where F_E' denotes the relative complement of F_E. Members of $\tilde{\tau}(\tilde{c})$ are the soft open sets of $(X, \tilde{c}, E)$ and their complements are the soft closed sets.

Definition 2.7. Let $(X, \tilde{c}, E)$ be a Čech soft closure space. If $\tilde{c}$ (F_E) = F_E for every soft set F_E contained in $\tilde{X}$, $\tilde{c}$ is called *the soft discrete closure operator* on X. If $\tilde{c}$ (F_E) = $\tilde{X}$, $\tilde{c}$ is called *the trivial Čech soft closure operator* on X.

1. Soft Separation Axioms

Definition 3.1. A Čech soft closure space $(X, \tilde{c}, E)$ is said to be *soft T_0* if for every $x \neq y$ in X either $x \notin \tilde{c}(y_E)$ or $y \notin \tilde{c}(x_E)$.

Example 3.2. Every discrete Čech soft closure spaces $(X, \tilde{c}, E)$ is a soft T_0-space because there exists a soft closed set x_E containing x but not containing y which is distinct from x.

Theorem 3.3. Every soft subspace of a soft T_0-space is a soft T_0-space.

Proof. Let $(X, \tilde{c}, E)$ be a Čech soft closure space and $(Y, \tilde{c}_Y, E)$ be a soft subspace $(X, \tilde{c}, E)$ of where $\tilde{c}_Y$ is $\tilde{c}$ -relative Čech soft closure space. Let x, y be two distinct points of Y and as $Y \subset X$ therefore these two are distinct points of X. Now $(X, \tilde{c}, E)$ is a soft T_0-space therefore either $x \notin \tilde{c}(y_E)$ or $y \notin \tilde{c}(x_E)$. Suppose that $x \in \tilde{c}(y_E)$ and $y \notin \tilde{c}(x_E)$ .Then by definition of $\tilde{c}_Y$, $\tilde{c}_Y(y_E) = \tilde{Y} \tilde{\cap} \tilde{c}(y_E)$ which contains x but does not contain y. Hence $(Y, \tilde{c}_Y, E)$ is also a soft T_0-space.

Theorem 3.4. A soft topological space $(X, \tilde{\tau}(\tilde{c}), E)$ under its Čech soft closure operator $\tilde{c}$ is a soft T_0-space if and only if the soft closures of distinct points are distinct.

Proof. Let x and y be two distinct points of X. Since $(X, \tilde{\tau}(\tilde{c}), E)$ is a soft T_0-space there exists a soft open set F_G such that $x \in F_G$ but $y \notin F_G$. Now F_G being soft open implies that $\tilde{X} - F_G$ is soft closed, $x \notin \tilde{X} - F_G$ and $y \in \tilde{X} - F_G$. Now by definition of soft closure of $\overline{y}_E$ which is the intersection of all soft closed sets containing y. Hence $y \in \overline{y}_E$ but $y \in \overline{y}_E$ as $x \notin \tilde{X} - F_G$. Therefore $\overline{x}_E$ and $\overline{y}_E$ are distinct.

Conversely, let x and y be two distinct points of X and $\overline{x}_E$ and $\overline{y}_E$ are also distinct. This implies that there exists at least one point $z \in X$ such that $z \in \overline{x}_E$ but $z \notin \overline{y}_E$. We claim that $x \notin \overline{y}_E$ because if $x \in \overline{y}_E$ then $x_E \tilde{\subseteq} \overline{y}_E$ implies that $\overline{x}_E \tilde{\subseteq} \overline{\overline{y}}_E$ so that $\overline{x}_E \tilde{\subseteq} \overline{y}_E$. Therefore $z \in \overline{x}_E$ implies $z \in \overline{y}_E$, which is a contradiction to our assumption that $z \notin \overline{y}_E$. Now $x \notin \overline{y}_E$ implies that $x \in \overline{y}_E^c$ which is soft open as $\overline{y}_E$ is soft closed. Thus $\overline{y}_E^c$ is soft open set containing x but not y. Hence $(X, \tilde{\tau}(\tilde{c}), E)$ is a soft T_0-space.

Theorem 3.5. If $(X, \tilde{c}, E)$ is a Čech soft closure space, cl is the soft closure operation in the associated soft topological space (here closure of a soft set we use the notation $\overline{F}_A$ instead of cl(F_A)), then $\tilde{c}$ is coarser than cl.

Proof. For any soft set F_A, we have $\tilde{c}(F_A) \tilde{\subseteq} \tilde{c}(\overline{F}_A)$, since $F_A \tilde{\subseteq} \overline{F}_A$. Also since $\overline{F}_A$ is closed in the associated soft topology, $\tilde{c}(\overline{F}_A) = \overline{F}_A$.

Therefore $\tilde{c}(F_A) \tilde{\subseteq} \overline{F}_A$.

Theorem 3.6. If $(X, \tilde{\tau}(\tilde{c}), E)$ is soft T_0-space then $(X, \tilde{c}, E)$ is soft T_0-space.

Proof. Let $(X, \tilde{\tau}(\tilde{c}), E)$ be soft T_0. If $x \neq y$ in X, then by theorem 3. 4, either $x \notin \overline{y}_E$ or $y \notin \overline{x}_E$. But $\tilde{c}(F_A) \tilde{\subseteq} \overline{F}_A$ for every $F_A \tilde{\subseteq} \tilde{X}$. So we get $x \notin \tilde{c}(y_E)$ or $y \notin \tilde{c}(x_E)$.

The following example shows that the converse of the theorem 3.6 is not true.

Example 3.7. Let X = {a, b}, E = {e_1, e_2}. Let be defined on X such that

$\tilde{c}(\{(e_1,\{a\})\}) = \{(e_1,\{a\})\}$,

$\tilde{c}(\{(e_1,\{b\})\}) = \{(e_1,\{b\})\}$,

$\tilde{c}(\{(e_2,\{a\})\}) = \{(e_2,\{a\})\}$,

$\tilde{c}(\{(e_2,\{b\})\}) = \{(e_2,\{b\})\}$,

$\tilde{c}(\{(e_1,\{a,b\})\}) = \{(e_1,\{a,b\})\}$,

$\tilde{c}(\{(e_2,\{a,b\})\}) = \{(e_2,\{a,b\})\}$, and for all other soft sets F_{Ei} over X, let

$$\tilde{c}(F_{E_i}) = \begin{cases} \phi, \text{if} F_{E_i} = \phi, \\ \tilde{\cup}\{\tilde{c}(e_i, F(e_i)) : (e_i, F(e_i)) \in F_{E_i}\} \text{otherwise} \end{cases}.$$

$\tilde{c}$ is the Čech soft closure operator on X. Here $(X, \tilde{c}, E)$ is soft T_0. But $(X, \tilde{\tau}(\tilde{c}), E)$ is the indiscrete soft topology which is not soft T_0.

Definition 3.8 A Čech soft closure space is said to be soft T_1 if for $x \neq y$ in X we have $x \notin \tilde{c}(y_E)$ and $y \notin \tilde{c}(x_E)$.

Example 3.10. Every discrete Čech soft closure space is soft T_1.

Theorem 3.9. Every soft subspace of a soft T_1-space is soft T_1.

Proof. Let $(X, \tilde{c}, E)$ be a Čech soft closure space which is soft T_1 and $(Y, \tilde{c}_Y, E)$ be a soft subspace of $(X, \tilde{c}, E)$ where $\tilde{c}_Y$ is $\tilde{c}$ – relative Čech soft closure space. Let x and y be two distinct points of Y and as $Y \subset X$ therefore these two are distinct points of X. Now $(X, \tilde{c}, E)$ is a soft T_1-space therefore $x \notin \tilde{c}(y_E)$ and $y \notin \tilde{c}(x_E)$. Then by definition of $\tilde{c}_Y$, $\tilde{c}_Y(x_E) = \tilde{Y} \tilde{\cap} \tilde{c}(x_E)$ which contains x but does not contain y and $\tilde{c}_Y(y_E) = \tilde{Y} \tilde{\cap} \tilde{c}(y_E)$ which contains y but does not contain x. Hence $(Y, \tilde{c}_Y, E)$ is also a soft T_1-space.

Theorem 3.10. For a Čech soft closure space $(X, \tilde{c}, E)$ the following are equivalent.

1. The Čech soft closure space $(X, \tilde{c}, E)$ is soft T_1.
2. For any $x \in X$, the soft set x_E is soft closed.
3. Every finite soft subset over X is soft closed.

Proof. $(1) \Rightarrow (2)$ Let $(X, \tilde{c}, E)$ be soft T_1. If possible, suppose x_E is not soft closed. That is, $\tilde{c}(x_E) \neq x_E$. So there exists $y \neq x, y \in \tilde{c}(x_E)$. But this contradicts the fact that $(X, \tilde{c}, E)$ is soft T_1. Therefore x_E is soft closed. $(2) \Rightarrow (3)$ For any $x \in X$, the soft set x_E is soft closed. Since finite union of soft closed sets is soft closed, every finite soft subset over X is soft closed. $(3) \Rightarrow (2)$ The implification is trivial. $(2) \Rightarrow (1)$ the soft sets x_E and y_E are soft closed, we have $\tilde{c}(x_E) = x_E, \tilde{c}(y_E) = y_E$ and so $x \notin \tilde{c}(y_E)$ and $y \notin \tilde{c}(x_E)$. Therefore $(X, \tilde{c}, E)$ is soft T_1.

Corollary 3.11. $(X, \tilde{c}, E)$ is soft T_1 if and only if is

$(X, \tilde{\tau}(\tilde{c}), E)$ soft T_1.

Theorem 3.12. Every soft T_1-space is soft T_0-space.

Proof. The proof is trivial.

The example 3.7, shows that the soft T_0-space need not imply soft T_1-space.

Conclusion. In this paper, we introduced and studied the soft separation axioms such as soft T_0-space and soft T_1-space in Čech soft closure spaces and also we discussed the relation between separation properties in Čech soft closure space $(X, \tilde{c}, E)$ and those in the associated soft topological space $(X, \tilde{\tau}(\tilde{c}), E)$.

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