GENERALIZING THE IDEAS OF NUMERICAL ANALYSIS TO TOPOLOGICAL VECTOR SPACES

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Abstract: In this paper ideas of numerical analysis are generalized to topological vector space and found an interpolation formula for a continuous function F from the field R of real numbers with the usual metric topology on it into a topological vector space V over R.

Keywords: Numerical Method, Metric topology, Topological vector space V over R.

Introduction: We know that the numerical methods are used in Mathematics when analytical methods are not available or when only approximate solution is acceptable. Numerical analysis is the analysis of these methods. It involves the development and evolution of methods for computing required numerical results from given numerical data.

Topological Vector Space over the real field R of real numbers:

Definition:

Let V be a vector space over the real field R of real numbers.

Let τ be a topology on V, such that

1. (a , b) $\rightarrow$ a-b is a continuous function from (V x V, x) into (V, τ) and

2. $\alpha \in$ R , a $\in$ V , (α , a) $\rightarrow \alpha$ a is a continuous function from (R x V, τ_1 xτ) into (V,τ) whereτ_1 is the usual topology on R .

Then (V,τ) is called a topological vector space over the field R.

2. Defining forward differences: Let (R, τ_1) be the field of real numbers R equipped with usual metric topology τ_1. Let (V(R), τ_2) be nay topological vector space V over the field of real numbers R, being topology of this topological vector space.

Let x0, x1, x2, ………xi,……. , xn be (n+1) equidistant, distinct points of R, written in the ascending order where

$x_i – x_{i-1} = h$, i = 1, 2, 3, …… , n .

Let, F be a continuous functions from

(R, τ_1) $\rightarrow$ (V(R), τ_2)

Whose values f_0, f_1, f_2, ……… f_i,……. , f_n are known only at points x_0, x_1, ……… x_i,……. , x_n . Such a function we know is called a tabular function, Thus (x_0, f_0), (x_1, f_1),…….. (x_i, f_i), ……..(x_n, f_n) are tabulated points of continuous function F. Suppose we wish to know the value of F at x such that $x_0 \leq x \leq x_1$. In this case we can derive a formula identical to Newton's forward interpolation formula.

For this we define forward differences for elements of V as Δf_i, = $f_{i+1} - f_i$, i=0,1,2, ………,n-1 is first forward difference at f_i.

In general for any positive integer k, 1 $\leq$ k $\leq$ n $\Delta^k f_i = \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i$

The highest forward difference that we will get for (n+1) tabulated values (x_0, f_0), (x_1, f_1),…….., (x_n, f_n) is the n^{th} which will occur at f_0. All these forward differences themselves will be elements of V. We construct the forward difference table as for a continuous mapping F : R → R (the usual case).

Forward difference table for images in a general topological vector space V over R:-

x	F(x) = f	Δ	$\Delta^2 f$	--	$\Delta^n f$
x_0	f_0				
		$\Delta f_0 = f_1 - f_0$	$\Delta^2 f_0 = \Delta f_1 - \Delta f_0$		
x_1	f_1				
		$\Delta f_1 = f_2 - f_1$	$\Delta^2 f_1 = \Delta f_2 - \Delta f_1$		$\Delta^n f_0$
x_2	f_2				
		$\Delta f_2 = f_3 - f_2$			
x_3	f_3				
-	-				
-	-				
x_i	f_i				
		$\Delta f_i = f_{i+1} - f_i$			
x_{i+1}	f_{i+1}				
-	-				
-	-				

x_{n-2}	f_{n-2}				
		$\Delta f_{n-2} = f_{n-1} - f_{n-2}$			
x_{n-1}	f_{n-1}				
		$\Delta f_{n-1} = f_n - f_{n-1}$			
x_n	f_n				

Representing the value of a continuous function F: (R,) → (V(R),) by a polynomial in V.

Let,

f = F(x) = a0 + a1(x-x0) + a2 (x-x0) (x-x1) + a3(x-x0) (x-x1) (x-x2) + …….. + an (x-x0) (x-x1) …… (x-xn-1) → (1)

For x0 ≤ x ≤ x1

where a0, a1, a2, a3, ….., ai,……., an are elements of V so that the whole of the right hand side of equation (1) is an element of V. In representing our continuous function F by the polynomial on R.H.S. of (1), we are assuming that

(i) our polynomial function

P(x) = a0 + a1(x-x0) + a2 (x-x0) (x-x1) +…….. + ai(x-x0) (x-x1) ……..(x-xi-1) +…..+ an (x-x0) (x-x1) …… (x-xn-1) is a continuous function from (R,) into a general topological vector space (V(R),) where ai in V, i = 0, 1, 2, ….., n and x R, xi R, i=0,1,2,……n and

(ii) The Stone-Weierstrass Theorem [9] is true for continuous function F from [x0, xn] into a general topological vector space (V(R), τ_2).

In the next two articles we investigate the conditions under which assumption (i) and (ii) above will be true.

Continuity of a polynomial function.

Theorem 1 :-

Let (x_0, f_0), (x_1, f_1), ……. (x_i,f_i)…….(x_n,f_n) be n +1 tabulated points of a continuous function F from (R, τ_1) into a topological vector space (V(R), τ_2) over the field R of real numbers. Then the polynomial function.

$P(x) = a_0 + a_1(x-x_0) + a_2 (x-x_0) (x-x_1) +…….. + a_i(x-x_0) (x_1) …..(x-x_{i-1}) +…..+ a_n (x-x_0) (x-x_1) …… (x-x_{n-1})$

where$a_i \in V$, i = 0,1,2, ….n $x \in R$, $x_i \in R$, i = 0,1,2,…….,n is a continuous function from $[x_0, x_n]$ into (V(R), τ_2)

Proof: -We know a function of the type $x \to (x-x_i)$, where x_i is a real constant is a continuous function from (R, τ_1) into (R, τ_1) and also product of a finite number of continuous functions from (R, τ_1) into (R, τ_1) is a continuous function.

Therefore,

$F_i : x \to y_i=(x-x_0) (x-x_i) ……. (x-x_{i-1})$

i = 1, 2, 3, ……. n are all continuous function from (R, τ_1) → (R, τ_1) → (1)

Also the functions

G_i : (R, τ_1) → (RxV, τ_1x τ_2) such that

$G_i(y_i)=(y_i,a_i)$ where a_i is a fixed element of V are continuous functions for

i = 1, 2,….. n → (2)

Also since (V(R), τ_2) is topological vector space, scalar multiplication is a continuous mapping from (RxV, τ_1x τ_2) into (V(R), τ_2)

Thus each of the mappings,

H_i :(RxV, τ_1x τ_2) into (V(R), τ_2)

Such that, $H_i (y_i a_i)=y_i a_i = a_i y_i$, i = 1,2, …..n

Where $a_i \in V$ is fixed is continuous mapping. → (3)

(1), (2) and (3) imply that the composite mappings

$I_i : (H_{i\,0}\, G_{i\,0}\, F_i)$: (R, τ_1) into (V(R), τ_2)

Such that,

$x \to a_i y_i = a_i (x-x_0) (x-x_1)……(x-x_{i-1})$ are continuous mappings,

i = 1,2,…..n. → (4)

Also the mapping,

I_o : (R, τ_1) → (V(R), τ_2)

such that $I_o (x) = a_0$

where a_0 in V is a constant vector, is a continuous mapping → (5)

Now let I_1, I_2 be two continuous mappings from (R, τ_1) → (V(R), τ_2)

Then the mapping from (R, τ_1) into (VxV, τ_1x τ_2) such that $X \to (I_1(x), I_2(x))$ is a continuous mapping. → (6)

But (V(R), τ_2) is a topological vector space implies that $(I_1(x), I_2(x)) \to I_1(x) + I_2(x)$ is a continuous mapping from (VxV, τ_1x τ_2) → (V(R), τ_2) → (7)

(6) and (7) implies that the composite mapping from (R, τ_1) → (V(R), τ_2) such that,

$x \to I_1(x) + I_2(x) = (I_1 +I_2) (x)$ is a continuous mapping.

In other words sum of two continuous mappings from (R, τ_1) into (V(R), τ_2) is a continuous mapping.

Extending this result to a finite number of continuous mappings from (R, τ_1) into (V(R), τ_2)

we get $I = I_0 + I_1 + I_2 + …..+I_n$ is continuous mapping from (R, τ_1) intoz(V(R), τ_2), where I_o is mapping as at (5) and I_i, i = 1,2,…..,n are the mappings as at (4).

Thus, I from (R, τ_1) into (V(R), τ_2) such that,

$I (x) = I_0(x) + I_1(x) + I_2(x)+…..+ I_i(x) + ….+ I_n(x) =a_0+a_1(x-x_0)a_2(x-x_0)(x- x_1)+……+a_i(x-x_0)(x-x_1)…..(x-x_{i-1})+…..+ a_n(x-_0) (x- x_1)…(x-x_{n-1})$ is a continuous function.

But this I is nothing but the polynomial p(x) in the statement of our theorem.

Generalization of Stone – Weierstrass theorem: Let [a,b] be an interval of the real line equipped with the induced topology from (R, τ_1),τ_1 being the usual metric topology on R. Let F be a continuous function from [a,b]into(V(R), τ_2), which is a finite dimensional topological vector space of dimension m, over the field R of real numbers.

Let $\{x_1, x_2, \ldots x_i \ldots x_m\}$ be a basis of V(R) and let I from V(R) into R^m(R) be the isomorphism. (Refer, [4])

such that,

$X = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_i x_i + \ldots + \alpha_m x_m \in V(R)$,

$\alpha_i \in R$, i = 1,2…..m

$I(x) = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_j e_j + \ldots + \alpha_m e_m$

where $\{e_1, e_2, e_3 \ldots e_j \ldots, e_m\}$ is standard basis of R^m(R). Further suppose that I is a continuous function from (V(R), τ_2) into (R^m (R),τ_3), where τ_3 is the usual metric topology on R^m. Then there exist a sequence of polynomials Qn(x) in V such that

$F(x) = \lim_{n\to\infty} Q_n(x)$ uniformly on [a,b].

Here $Q_n(x)$ are polynomials of the type

$Q_n(x) = C_0 + C_1x + C_2x^2 + \ldots + C_ix^i + \ldots + C_nx^n$, $C_i \in V$, i = 0, 1, 2, 3…..n, $x \in R$, $x^i \in R$, i=0, 1, 2, 3, ….., n.

Proof: F from ([a,b], τ/[a,b]) into (V(R), τ_2) is a continuous function and I from (V(R), τ_2) onto (R^m (R),τ_3) is a continuous function implies $I^oF = G$ is a continuous function from [a,b] into (R^m (R),τ_3)

→ (1)

Let $g_1, g_2, \ldots g_3 \ldots, g_j, \ldots g_m$ be the component functions of G.

Thus $a \leq x \leq b$, $G(x) = (g_1(x), g_2(x), \ldots g_j(x), \ldots g_m(x))$ → (2)

Eq. (1) implies that $g_j : [a,b] \to R$ are continuous functions j= 1, 2, 3, ….., m

Thus applying the Stone-Weirerstrass theorem [9] , to g_j we know there exist a sequence of polynomials $p_{n,j}$, n= 1, 2, 3, …… , such that, $g_j(x) = \lim_{n\to\infty} p_{n,j}(x)$

uniformly on

[a,b], j= 1,2,…..m. → (3)

Now $a \leq x \leq b$,

G (x) = I (F (x))

Thus $F(x) = I^{-1}(G(x)) = I^{-1}(g_1(x), g_2(x), \ldots g_j(x), \ldots g_m(g))$ /

by equation (2))

$$= I^{-1}\begin{pmatrix} \lim_{n\to\infty} P_{n,1}(X), \lim_{n\to\infty} P_{n,2}(X), \\ \ldots \lim_{n\to\infty} P_{n,j}(X), \ldots \lim_{n\to\infty} P_{n,m}(X) \end{pmatrix}$$

(by equation (3))

$= \lim_{n\to\infty} I^{-1}(P_{n,1}(x), P_{n,2}(x), \ldots, P_{n,j}(x), \ldots P_{n,m}(x))$

→ (4)

Let,

$P_{n,1}(x) = a_{01} + a_{11}x + a_{21}x^2 + \ldots + a_{n1}x^n$

..

..

..

$P_{n,2}(x) = a_{02} + a_{12}x + a_{22}x^2 + \ldots + a_{n2}x^n$

$P_{n,j}(x) = a_{0j} + a_{1j}x + a_{2j}x^2 + \ldots + a_{n,j}x^n$

..

..

..

$P_{n,m}(x) = a_{0m} + a_{1m}x + a_{2m}x^2 + \ldots + a_{nm}x^n$

$a_{ij} \in R$, i = 0, 1, 2, …… n, & j = 1, 2, ……, m

Thus $(P_{n,1}(x), P_{n,2}(x), \ldots, P_{n,j}(x), \ldots, P_{n,m}(x)) = P_{n,1}(x)e_1 + P_{n,2}(x)e_2 + \ldots + P_{n,j}(x)e_j + \ldots + P_{n,m}(x)e_m$

$= (a_{01} + a_{11}x + a_{21}x^2 + \ldots a_{n1}x^n)e_1$

$+(a_{02} + a_{12}x + a_{22}x^2 + \ldots a_{n2}x^n)e_2$

…

…

$+ (a_{0j} + a_{1j}x + a_{2j}x^2 + \ldots a_{nj}x^n)e_j$

…

…

$+ (a_{0m} + a_{1m}x + a_{2m}x^2 + \ldots a_{nm}x^n)e_m$

$= (a_{01}e_1 + a_{02}e_2 + \ldots + a_{0j}e_j + \ldots + a_{om}e_m)$

$+(a_{11}e_1 + a_{12}e_2 + \ldots + a_{1j}e_j + \ldots + a_{1m}e_m)x$

$+(a_{21}e_1 + a_{22}e_2 + \ldots + a_{2j}e_j + \ldots + a_{im}e_m)x^2$

…

…

$+(a_{i1}e_1 + a_{i2}e_2 + \ldots + a_{ij}e_j + \ldots + a_{im}e_m)x^i$

…

…

$+(a_{n1}e_1 + a_{n2}e_2 + \ldots + a_{nj}e_j + \ldots + a_{nm}e_m)x^n$

$= b_0 + b_1x + b_2x^2 + \ldots + b_ix^i + \ldots + b_nx^n$

where, $b_i = a_{i1}e_1 + a_{i2}e_2 + \ldots + a_{ij}e_j + \ldots + a_{im}e_m$

i = 0, 1, 2, …. , n and j = 1, 2, ….., m,

b_i, i = 0, 1, 2, ….. n are vectors in R^m. → (5)

I is an isomorphism from V(R) onto R^m (R) implies

I^{-1} is an isomorphism from R^m (R) ontoV(R).

→ (6)

Thus (4) and (5) imply

$F(x) = \lim_{n\to\infty} I^{-1}(b_0 + b_1x + b_2x^2 + \ldots + b_ix^i + \ldots + b_nx^n)$

$= \lim_{n\to\infty} I^{-1}(b_0 + xb_1 + x^2b_2 + \ldots + x^ib_i + \ldots + x^nb_n)$

$= \lim_{n\to\infty} \{I^{-1}(b_0) + I^{-1}(xb_1) + I^{-1}(x^2b_2) + \ldots + I^{-1}(x^ib_i) + \ldots + I^{-1}(x^nb_n)\}$

[By equation (6)]

$= \lim_{n\to\infty} \{I^{-1}(b_0) + xI^{-1}(b_1) + x^2I^{-1}(b_2) + \ldots + x^iI^{-1}(b_i) + \ldots + x^nI^{-1}(b_n)\}$

Again by equation (6)

$F(x) = \lim_{n\to\infty} \{I^{-1}(b_0) + I^{-1}(b_1)x + I^{-1}(b_2)x^2 + \ldots +$

$I^{-1}(b_i)x^i + \ldots + I^{-1}(b_n)x^n\}$

$\rightarrow$ (7)

Let $I^{-1}(b_i) = C_i$, i = 0,1,2, ……, n.

Then it follows from (6) that C_i are in V, i = 0,1,2, ….n.

Thus (7) can be rewritten as

$F(x)=\lim_{n\to\infty} (C_0+C_1x + C_2x^2 + \ldots + C_ix^i + \ldots + C_nx^n$

$= \lim_{n\to\infty} Q_0(x) \rightarrow (8)$

Now x in [a,b],

$G(x) = (g_1(x), g_2(x),\ldots,g_j(x),\ldots, g_m(x))$

$$=\left(\lim_{n\to\infty} P_{n,1}(x), \lim_{n\to\infty} P_{n,2}(x),\ldots, \lim_{n\to\infty} P_{n,j}(x),\ldots, \lim_{n\to\infty} P_{nm}(x)\right)$$

$= \lim_{n\to\infty} (P_{n,1}(x), P_{n,2}(x),\ldots, P_{n,j}(x),\ldots, P_{nm}(x))$

uniformly on [a,b]. (by eq. (3))

Thus $F(x) = I^{-1}(G(x))$

$= I^{-1}\lim_{n\to\infty} (P_{n,1}(x), P_{n,2}(x),\ldots, P_{n,j}(x),\ldots,$

$P_{nm}(x))$

$= \lim_{n\to\infty} I^{-1} (P_{n,1}(x), P_{n,2}(x),\ldots, P_{n,j}(x),\ldots,$

$P_{nm}(x))$ uniformly on [a,b]. $\rightarrow$ (9)

But we have proved that the right hand sides of (8) and (9) are the same. Thus the convergence at (8) is uniform on [a,b].

This proves the theorem.

Derivation of the forward interpolation formula: Having proved the theorems at 2.3 and 2.4 we are justified in representing a continuous tabulated function F as at 2.1, 2.2 by a polynomial

$a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+\ldots+a_i(x-x_0)(x-x_1)\ldots(x-x_{i-1}) +\ldots$

$+a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1})$

Where ai $\in$ V, i = 0, 1, 2, ….. , n and x $\in$ R

etc. for sufficiently high value of n.

Thus,

$F(x) = a_o+a_1(x-x_o)+a_2(x-x_o)(x-x_1)+\ldots+a_i(x-x_o)(x-x_1)\ldots(x-x_{i-1}) +\ldots +a_n(x-x_o)(x-x_1)\ldots(x-x_{n-1})$

$\rightarrow$ (I)

Putting $x = x_o$ in (I) we get $a_o = f_o$

Putting $x = x_1$ in (I) we get

$f_1 = F(x_1) = a_o + a_1 (x_1-x_o) = f_o + a_1h$

Thus $a_1 = \frac{f_1-f_0}{h} = \frac{\Delta f_0}{h} = \frac{\Delta f_0}{1!h}$

Substituting $x = x_2$ in (I) we get,

$f_2 = F(x_2) = a_o + a_1 (x_2-x_o) = a_2 (x_2-x_o) (x_2-x_1)$

$= f_o + \frac{\Delta f_0 . 2h}{h} + a_2 . 2h .h$

Thus,

$a_2 = \frac{1}{2h^2}\{f_2- f_o- 2 \Delta f_o\}$

$= \frac{1}{2!h^2}\{f_2- f_o- 2 (f_1- f_o)\}$

$= \frac{1}{2!h^2}\{(f_2- f_1)- (f_1- f_o)\}$

$= \frac{1}{2!h^2}\{\Delta f_1- \Delta f_o\}$

$= \frac{1}{2!h^2}\Delta^2 f_0$

Proceeding in this manner we get,

$a_i = \frac{1}{i!h^i}\Delta^i f_0$

-

-

$a_n = \frac{1}{n!h^n}\Delta^n f_0$

Thus, $F(x) = f0 + \frac{\Delta f_0}{1!h}(x-x_0) + \frac{\Delta^2 f_0}{2!h^2}(x-x_0)(x-x_1) + \ldots$

$+ \frac{\Delta^n f_0}{n!h^n}(x-x_0)(x-x_1)\ldots(x-x_{n-1}) \rightarrow$ (II)

An Application:-

Let W(R) = vector space over R of continuous real – valued functions on [0,1].

Let V(R) = Subspace of W(R) consisting of real – valued constant function on [0,1]. We know V(R) is a finite dimensional vector space over R of dimension 1.

For F $\in$ V(R) , F = CI

where I : [0,1] $\rightarrow$ R such that , x $\in$ [0,1], I(x) = 1 and C $\in$ R such that x $\in$ [0,1] , F(x) = C.

We think of V(R) as a vector space in its own right of dimension 1 over R and define a tabular function

F : R $\rightarrow$ V (R)

as $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3$

$F(x_0) = f_0, F(x_1) = f_1, F(x_2) = f_2, F(x_3) = f_3$

Where, $f_0 (x) = 2$, x $\in$ [0,1]

$f_1 (x) = 4$, x $\in$ [0,1]

$f_2 (x) = 5$, x $\in$ [0,1]

$f_3 (x) = 8$, x $\in$ [0,1].

Assuming F is continuous when V(R) is a topological vector space with respect to a certain topology τ on V(R) we want to find value F at x = 0.5

Our forward difference table is

Assuming F is continuous when V(R) is a topological vector space with respect to a certain topology τ on V(R) we want to find value F at x = 0.5

Our forward difference table is

x	F (x) = f	Δf	Δ^2f	Δ^3f
$x_0 = 0$	f_0 (x)= 2			
		Δf_0 (x)= 2		
$x_1 = 1$	f_1 (x)= 4		$\Delta^2 f_0$ (x)= -1	
		Δf_1 (x)= 1		$\Delta^3 f_0$ = 3
$x_2 = 2$	f_2 (x)= 5		$\Delta^2 f_1$ (x)= 2	
		Δf_2 (x)= 3		
$x_3 = 3$	f_3 (x)= 8			

Thus, using formula (II) of 2.5 above we get, (Here h=1)

$f = F(0.5) = f_0 + \frac{\Delta f_0}{1!}(0.5-0) + \frac{\Delta^2 f_0}{2!}(0.5-0)(0.5-1) + \frac{\Delta^3 f_0}{3!}(0.5-0)(0.5-1)(0.5-2) = f_0 + 0.5\,\Delta f_0 + \frac{(0.5)(-0.5)}{2}\Delta^2 f_0 + \frac{(0.5)(-0.5)(-3/2)}{6}\Delta^3 f_0$

$= f_0 + \frac{\Delta f_0}{2} - \frac{\Delta^2 f_0}{8} + \frac{\Delta^3 f_0}{16}$

Thus, F maps x = 0.5 into that constant real valued function f on [0,1] such that , x ∈ [0,1]

$f(x) = f_0(x) + \left(\frac{\Delta f_0}{2}\right)(x) - \left(\frac{\Delta^2 f_0}{8}\right)(x) + \left(\frac{\Delta^3 f_0}{16}\right)(x)$

$= 2 + \frac{2}{2} - \frac{(-1)}{8} + \frac{3}{16}$

$= \frac{53}{16} = 3.3125$

Conclusion: The main aim of this paper is to generalize the ideas of Numerical Analysis to topological vector spaces which is very necessary to connect two branches of mathematics.

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