SOME RESULTS ON REVERSE DERIVATIONS IN PRIME RINGS

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Abstract: In this paper, we prove some results on reverse derivations in prime rings. We prove that let R be a prime ring of char. $\neq 2$ and d_1, d_2 reverse derivations of R such that the iterate d_1d_2 is also a reverse derivation, then one at least of d_1, d_2 is zero.

Keywords: Derivation, Reverse derivation, Prime ring, Center.

Introduction: Posner[5], Herstein[4], Felzenszwalb[3], Daif and Bell[2] have investigated the properties of prime (or) semi prime rings with derivations. Bresar and Vukman[1] have introduced the notion of a reverse derivation and Samman and Alyamani [6] have studied some properties of prime (or) semi prime rings with reverse derivations. In this paper, we prove some results on reverse derivations in prime rings.

Preliminaries: We know that an additive map d from a ring R to R is called a derivation on R, if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A ring R is called prime if $xay = 0$ implies $x = 0$ or $y = 0$ for all x, a, y in R. An additive mapping d from a ring R into itself satisfying $d(xy) = d(y)x + yd(x),$ for all $x, y \in R,$ is called a reverse derivation on $R.$ Throughout this paper R will denote a prime ring and Z its center. We use the following elementary identities in this paper.

In any ring R, $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$ hold for all $x, y, z \in R$.

If R is any ring and d any derivation, we have $d(Z) \subseteq Z, d([x, y]) = [x, d(y)] + [d(x), y]$.

Main results:

Theorem 1: Let R be any ring, d be a reverse derivation of R such that $d^3 \neq 0$. Then A, the subring generated by all $d(r)$, $r \in R$, contains a non-zero ideal of R.

Proof: Since $d^3 \neq 0$ and $d(R) \subset A$, $d^2(A) \neq 0$.

We assume that $y \in A$ such that $d^2(y) \neq 0$.

If $x \in R$, then

$d(xy) \in A$ and $d(xy) = d(y)x + yd(x)$.

Since both y and $d(x)$ are in A, then $d(y)x \in A$ which implies $d(y)R \subset A$.

Similarly, $Rd(y) \subset A$.

If $r, s \in R$, then

$d(sd(y)r) \in A$ and

$d(sd(y)r) = d(d(y)r)s + d(y)rd(s)$.

$d(sd(y)r) = d(r)d(y)s + rd^2(y)s + d(y)rd(s)$ But $d(y)r \in A$ and $d(y)s \in A$ which implies $rd^2(y)s \in A$, for all $r, s \in R$.

Since $Rd^2(y) \subset A$, $d^2(y)R \subset A$ and $Rd^2(y)R \subset A$, we have that the ideal of R generated by $d^2(y) \neq 0$ must be in A. This proves the theorem.

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Now, we present some results on reverse derivations in prime rings.

Lemma 1: Let d be a reverse derivation of a prime ring R and a be an element of R. If $ad(x) = 0$, for all $x \in R$, then either $a = 0$ (or) d is zero.

Proof: In $ad(x) = 0$, for all $x \in R$, we replace x by yx then,

$$\Rightarrow ad(yx) = 0$$
$$\Rightarrow ad(x)y + axd(y) = 0$$
$$\Rightarrow axd(y) = 0, \text{ for all } x, y \in R.$$

If d is not zero, that is, if $d(y) \neq 0$, for some $y \in R$, then by the definition of a prime ring, $a = 0$.

Theorem 2: Let R be a prime ring of char. $\neq 2$ and d_1, d_2 reverse derivations of R such that the iterate d_1d_2 is also a reverse derivation, then one at least of d_1, d_2 is zero.

Proof: It is given that d_1d_2 is a reverse derivation.

So, $$d_1d_2(ab) = d_1d_2(b)a + bd_1d_2(a)$$
$$= bd_1d_2(a) + ad_1d_2(b)$$

However d_1, d_2 are reverse derivations. So,

$$d_1d_2(ab) = d_1(d_2(ab))$$
$$= d_1(d_2(b)a + bd_2(a))$$
$$= d_1(d_2(b)a) + d_1(bd_2(a))$$

$$= d_1(a)d_2(b) + ad_1d_2(b) + bd_1d_2(a) + d_2(a)d_1(b)$$

But, $d_1d_2(ab) = bd_1d_2(a) + ad_1d_2(b)$, so,

$d_1(a)d_2(b)+d_2(a)d_1(b)=0$,for all $a,b\in R$.……………….. (1)

We replace a by $d_1(c)a$ in equ.(1), then

$d_1(d_1(c)a)d_2(b)+d_2(d_1(c)a)d_1(b)=0$,

for all $a,b,c\in R$.

$\Rightarrow d_1(a)d_1(c)d_2(b)+ad_1(d_1(c))d_2(b)+$

$d_2(a)d_1(c)d_1(b)+ad_2(d_1(c))d_1(b)=0$

Now, $a(d_1(d_1(c))d_2(b)+d_2(d_1(c))d_1(b))=0$,

Since $d_1(d_1(c))d_2(b)+d_2(d_1(c))d_1(b)=0$, which is merely equ.(1) in which a is replaced by $d_1(c)$. Then, we get, $d_1(a)d_1(c)d_2(b)+d_2(a)d_1(c)d_1(b)=0$, for all $a,b,c\in R$. ……………….. (2)

From equ. (1), we have,

$$d_1(a)d_2(b)=-d_2(a)d_1(b)$$

If a is replaced by c in the above equation, then we get,

$$d_1(c)d_2(b)=-d_2(c)d_1(b)$$

Now, equ.(2) becomes,

$$-d_1(a)d_2(c)d_1(b)+d_2(a)d_1(c)d_1(b)=0$$

By factoring out $d_1(b)$ on the right, we have,

$(d_2(a)d_1(c)-d_1(a)d_2(c))d_1(b)=0$,

for all $a,b,c\in R$.

By Lemma: 1, $d_2(a)d_1(c)-d_1(a)d_2(c)=0$, for all $a,c\in R$, unless d_1 is zero. …… (3)

We replace b by c in equ.(1), then we get,

$d_1(a)d_2(c)+d_2(a)d_1(c)=0$,

for all $a,c\in R$. ……………..(4)

By adding equ.'s (3) and (4), we get,

$d_2(a)d_1(c)-d_1(a)d_2(c)+d_1(a)d_2(c)+$

$d_2(a)d_1(c)=0$

$$\Rightarrow 2d_2(a)d_1(c)=0$$

$$\Rightarrow d_2(a)d_1(c)=0 ,$$

for all $a,c\in R$ or else d_1 is zero.

By using Lemma:1, again with replacing a by $d_2(a)$, we find that d_1 is zero or else $d_2(a)=0$ for all $a\in R$, i.e., $d_1=0$ (or) $d_2=0$.

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