NILPOTENCY IN WEAKLY STANDARD RINGS

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Abstract: In this paper, using the properties of the weakly standard ring, we prove that a left or right nilpotent weakly standard ring is nilpotent.

Keywords: weakly standard ring, left nucleus, middle nucleus, prime ring.

Introduction: In [1] Zelmanov showed that a left or right nilpotent alternative ring must be nilpotent. Pokrass [2] studied the solvability and nilpotency in generalized alternative rings. Smith [3] extended the result of Pokrass and proved that for flexible derivation alternator rings either left or right nilpotence implies nilpotence.In this paper, using the properties of the weakly standard ring we prove that a left or right nilpotent weakly standard ring is nilpotent. Throughout this paper R will denote a weakly standard ring.

Preliminaries: In a weakly standard ring we have

(x, y, x) = 0, (1)

((w,x,y),z) = 0 (2)

and ((x,y,y)z,w) = 0. (3)

We know that V = {v∈ R/(v,R) = 0 = (vR,R)} is an ideal of R.

If R is any ring and d any derivation, wehave, $d(Z) \subseteq Z, d([x,y]) = [x,d(y)] + [d(x),y]$

Main results: Now we prove an important identity in the following lemma.

Lemma 1: *If R is a simple weakly standard ring, then (yz, x, x) = y(z,x,x) + (y,x,x)z.*

Proof: From (2) and (3), we have

((yz,x,x) - y(z,x,x) - (y,x,x)z,r) = 0 and ({yz,x,x) - y(z,x,x) - (y,x,x)z}w,r) = 0.

So (yz,x,x) - y(z,x,x) - (y,x,x)z is in V.

Since R is simple and V is an ideal of R, either V = R or V = 0. If V = R, then R is commutative. But R is not commutative. Thus V=0 and (yz,x,x) - y(z,x,x) - (y,x,x)z = 0.

That is, (yz,x,x) = y(z,x,x) + (y,x,x)z. (4)

This proves the lemma.

For a∈ A, we denote by L_a and R_a the operators of left and right multiplication by a. the notation S_a will be used when an operator S can be either L or R. We note that, due to flexibility, when R satisfies some relation involving multiplication operators, it also satisfies the opposite relation where L's and R's are interchanged.

In linearized form, identities (1) and (4) are

(x,y,z) + (z,y,x) = 0 (5)

(yz,x,w) + (yz,w,x) = y{(z,x,w) + (z,w,x)}+{(y,x,w) + (y,w,x)}z. (6)

If we take as argument first x and then y, (5) is equivalent in operator form to each of the following:

$R_yR_z – L_yL_z= R_{yz} – L_{zy}$, (7)

$L_xR_z – R_z L_x + L_z R_x – R_xL_z = 0$. (8)

Similarly, taking in turn y,z and x as the argument, (6) can be

$R_z R_xR_w + R_zR_w R_x – R_xR_wR_z – R_wR_xR_z =$

$R_zR_{xw} + R_zR_{wx} - R_{xw}R_z - R_{wx}R_z + R_{(z,x,w)+(z,w,x)}$ (9)

$L_yR_xR_w + L_yR_wR_x - R_xR_wL_y - R_wR_xL_y =$

$L_yR_{xw} + L_yR_{wx} - R_{xw}L_y - R_{wx}L_y + L_{(y,x,w)+(y,w,x)}$,

(10)

$L_zR_wL_y - R_wL_zL_y - L_wL_zL_y + L_yR_wR_z - R_wL_yR_z - L_wL_yR_z =$

$-R_wL_{yz} - L_wL_{yz} + L_{yz}R_w - L_{zw}L_y - L_{yw}R_z + L_{(yz)w}$.

(11)

Let B denote the ideal A^2 of the ring R. For operators T and T^l on R, we shall write $T \equiv T^l$ if $T-T^l = \Sigma T_i$ where each operator Ti has a factor of the form S_{bi} with bi ∈ B.

Lemma 2: *If the operator $T=S_{x1} \dots S_{x2m}$, then $T \equiv T^l$ where T^l factors into m pairs of multiplication operators with each pair having atleast one L .*

Proof: By assumption T is itself the product of m pairs of multiplication operators. Each of these pairs either has an L or consists of two consecutive R's. But by (7), $R_yR_z = L_yL_z + R_{yz} - L_{zy} \equiv L_yL_z$, since yz, zy∈ A^2 = B. Thus substituting for any double-R pairs in T, we obtain $T \equiv T^l$ where T^l has the required form.

Lemma 3 :*If the operator $T = S_{x_i} \dots\dots S_{x_J}$ where t≥1 of the S's are Ls, then $T = \sum L_{y_1} \dots\dots L_{y_{t-1}} S \dots . S$.*

Proof: For j = 1 or 2 the lemma is immediately true. Thus we assume j ≥ 3 and that the lemma holds for values less than j. Then for an operator T of the form LS… S, our induction assumption applied to the factor S..S completes the proof. Hence we are reduced to the case T = RS…S, where we shall consider the initial factor RSS of T. First using (7) to substitute for RR, we see RRS ≡ LLS. Next using (7) to substitute for LL, we have RLL ≡ RRR ≡ LLR since RRS ≡ LLS. Then (11) gives RLR≡ -RLL + ∑LSS ≡∑LSS, since RLL ≡ LLR. This shows that in all cases we can make substitutions reducing T = RS…S to the form T ≡∑LS….S. But as already noted, the induction assumption now applied to the factors S…S completes the proof.

Lemma 4 :*If the operator $T = S_{x_i} \dots\dots S_{x_n}$ is such that j of the $x_i \in B$ where 1≤j≤n, then $T = \sum S_{y_1} \dots\dots S_{y_{t_k}} S \dots . S$ where in each term atleast j of the $y_i \in B$ and for each $1 \le i < t_k$ either$y_i \in B$ or $y_{i+1} \in B$.*

Proof : We begin by showing that if $T = S_{x_1} \dots\dots S_{x_n} S_b$ where b∈ B and n ≥ 1, then T can be expressed as a sum of terms each of the form $S_{\underline{b}} S_{y_1} \ldots\ldots S_{y_k}$ or $S_{y_1} S_{\underline{b}} S_{y_2} \ldots\ldots S_{y_k}$ where $\underline{b} \in R$. The proof of this is by induction on n, with n =1 being immediate. Thus we let $T = S_{x_1} \ldots\ldots S_{x_{n-1}} S_{x_n} S_b$ where $n \geq 2$, and consider the eight possible forms for $S_{x_{n-1}} S_{x_n} S_b$.

Suppose first $S_{x_{n-1}} S_{x_n} S_b = R_{x_{n-1}} R_{x_n} R_b$. Then setting $z = x_{n-1}$, $x = x_n$, and $w = b$ in (9), we see

$$R_{x_{n-1}} R_{x_n} R_b = -R_{x_{n-1}} R_b R_{x_n} + R_{x_n} R_b R_{x_{n-1}} + R_b R_{x_n} R_{x_{n-1}} + R_{x_{n-1}} R_{x_n b} + R_{x_{n-1}} R_{b x_n} - R_{x_n b} R_{x_{n-1}} - R_{b_{x_n}} R_{x_{n-1}} + R_{(x_{n-1}, x_n, b)+(x_{n-1}, b, x_n)}.$$

Thus making this substitution for $R_{x_{n-1}} R_{x_n} R_b$ gives $T = \sum T^{l}$, where each T^{l} has an initial factor of the form $S_{y_1} S_{y_k} S_{\underline{b}}$ with $k < n$. If we now apply the induction assumption to these factors, then T is expressed in the required manner.

Suppose next $S_{x_{n-1}} S_{x_n} S_b = R_{x_{n-1}} L_{x_n} L_b$ or $L_{x_{n-1}} L_{x_n} R_b$. Using (7) to substitute for LL, both these cases reduce to $R_{x_{n-1}} R_{x_n} R_b + \sum S_{y_{n-1}} S_{\underline{b}}$. But $R_{x_{n-1}} R_{x_n} R_b$ was just established, and $\sum S_{y_{n-1}} S_{\underline{b}}$ follows from our induction. Finally, if $S_{x_{n-1}} S_{x_n} S_b = L_{x_{n-1}} R_{x_n} L_b$, we set $x = x_n$, $z = b$ in (8). This gives the substitution $R_{x_n} L_b = L_{x_n} R_b - R_b L_{x_n} + L_b R_{x_n}$, which using the induction assumption reduces to the established case $L_{x_{n-1}} L_{x_n} R_b$. Since going to the opposite ring gives the four remaining cases, this completes the induction.

We now prove the statement of the lemma by inducting on j, with our initial observation implying the case j = 1. Thus let $T = S \ldots . S_{b_1} \ldots . S_{b_{j-1}} \ldots S_{b_j} \ldots S$ where the $b_j \in B$. By induction we can express $S \ldots . S_{b_1} \ldots . S_{b_{j-1}}$ as a sum of terms of the form $S_{y_1} \ldots . S_{y_t} S \ldots S$, where each of these terms has the required property for j - 1. This means T can be expressed as a sum of terms having the form $S_{y_1} \ldots . S_{y_t} S \ldots S_{b_j} \ldots S$.

If $y_t \in B$, our initial observation can be applied to S_{b_j}. If $y_t \notin B$, then $y_{t-1} \in B$ and our initial observation can be applied to $S_{y_t} S \ldots S_{b_j}$. Thus T is now expressed as a sum of terms each having the required property for j, which completes the proof of the lemma.

The following lemma is proved in [3].

Lemma 5: *Let $A^{[2m]}=(0)$ where $m \geq 2$. If the operator $T = S_{x_i} \ldots . S_{x_{2m}}$ where for each $1 \leq i \leq 2m$ either $x_i \in B$ or $x_{i+1} \in B$, then* $(B^k)T \subseteq (B^{k+1}) \sum S \ldots . S$.

We use this lemma to prove the following theorem.

Theorem 1 : *If R is a left or right nilpotent weakly standard ring, then R is nilpotent.*

Proof : Suppose first that R is left nilpotent. Let $R^{(0)} = R$ and define inductively $R^{(k)} = \left(R^{(k-1)}\right)^2$. Then R is called solvable of index n if $R^{(n)} = (0)$ and n is the least such integer. It is immediate that any left nilpotent ring is solvable. Thus R is solvable, and to prove R is nilpotent we induct on the index of solvability of R. To start, R is clearly nilpotent when $R^2 = R^{(1)} = (0)$. Now by induction we can assume $B = R^2$ is nilpotent, since the ideal B is a left nilpotent weakly standard ring with solvable index one less than that of R. Thus let $B^n = (0)$, and by the left nilpotence of R let $R^{[2m]} = (0)$ where $m \geq 2$.

We shall show that for r = 2(2m + 1)(2m(n - 1)) + 1 each operator $T = S_{x_1} \ldots . S_{x_r}$ is zero, which by theorem 2.4 in [4] establishes that R itself is nilpotent. First we note $(R)T \subseteq (B) S_{x_2} \ldots . S_{x_r}$ Now $S_{x_2} \ldots . S_{x_r}$ is the product of 2m(n-1) factors each having length 2(2m + 1). By Lemma 2 we can express each of these factors of length 2(2m + 1) as $T^{l} + \sum T_i$, where T^{l} has 2m + 1 L's and each Ti has a factor of the form S_b. By Lemma 3, T^{l} can in turn be expressed as $\sum L_{y_1} \ldots L_{y_{2m}} S \ldots S + \sum T_i$. But since $R^{[2m]} = (0)$, any operator with 2m adjacent L's is zero. Thus after making these substitutions and multiplying, we arrive at $S_{x_2} \ldots S_{x_r} = \sum T_i^1$ where each T_i^{l} has 2m(n-1) factors of the form S_b.

Now each T_i^{l} is the product of n-1 factors each having 2m factors of the form S_b. In each T_i^{l} we consider the first of these factors with 2m S_b's. Using lemma 4, we can substitute for these initial factors to obtain $T_i^{l} = \sum T_1 T_2$, where $T_1 = S_{y_1} \ldots S_{y_t} S \ldots S$ with at least 2m of the $y_i \in B$ and for each $1 \leq i < t$ either $y_i \in B$ or $y_{i+1} \in B$, and where T_2 is the product of n-2 factors each having 2m factors of the form S_b. Thus by lemma 5, $(R)T \subseteq (B) S_{x_2} \ldots S_{x_r} = (B) \sum T_i^{l} = (B) \sum T_1 T_2 \subseteq (B^2) \sum S \ldots S T_2$.

We now consider in each T_2 the first factor with 2mS_b's, and repeat the preceding argument. Continuing in this fashion, one arrives at $(R)T \subseteq (B^n) \sum S \ldots S = (0)$, which shows R is nilpotent as claimed.

Finally, suppose R is right nilpotent. Then the opposite ring of R is a left nilpotent weakly standard ring. Thus the opposite ring is nilpotent. Hence R itself is nilpotent.

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