# SOME RESULTS ON WEAKLY STANDARD RINGS

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**Abstract:** In this paper we first show that a prime ring is weakly standard if and only if it is either associative or commutative. Then we prove that simple rings and primitive rings are prime, with the exception of rings R, such that $R^2 = 0$. At the end of this paper we give an example of a weakly standard ring which is not an accessible ring.

**Keywords:** weakly standard ring, left nucleus, middle nucleus, prime ring.

**Introduction:** The weakly standard rings were introduced by Sansouice [1]. He obtained some results of weakly standard rings which are parallel to those of Kleinfled for accessible rings, but the method is different. In [1] it is proved that simple flexible rings satisfying ((w,x),y,z) = 0 are either associative or commutative. This generalizes the results for weakly standard rings.

In this paper we first show that a prime ring is weakly standard if and only if it is either associative or commutative. Then we prove that simple rings and primitive rings are prime, with the exception of rings R, such that $R^2 = 0$. At the end of this paper we give an example of a weakly standard ring which is not an accessible ring. Throughout this paper R denotes a weakly standard ring.

**Preliminaries:** We know that a nonassociative ring R is a weakly standard ring, if it satisfies the following identities:

(x,y,x) = 0 (flexible), (1)

((w,x),y,z) = 0, (2)

And (w,(x,y),z) = 0 (3)

To facilitate our computations, we begin with the following identities of an arbitrary ring:

(wx,y,z) – (w,xy,z) + (w,x,yz) + (w,x,yz) = w(x,y,z) + (w,x,y)z, (4)

(xy,z) = x(y,z) + (x,z)y + (x,y,z) + (z,x,y) - (x,z,y) (5)

We now let $N_l = \{l \in R \mid (l, x, y) = 0\}$, $N_m = \{m \in R \mid (x,m,y) = 0\}$. Then we call $N_l$ the left nucleus and $N_m$ the middle nucleus of R.

With these definitions, the flexible ring R is weakly standard if and only if commutators are in $N_l \cap N_m$. For any $l \in N_l$, we have (lx, y,z) = l(x,y,z) while (1) implies that (x,y,l) = 0. A ring R is called prime if, whenever A and B are ideals of R such that AB = 0, then either A = 0 or B = 0. Our first goal is to prove that a prime ring is weakly standard if and only if it is either associative or commutative. We take a significant step in this direction by letting C be the set of all commutators of a weakly standard ring R and prove the following lemmas.

**Lemma 1:** *If A is the set of all finite sums of elements of the form R (C, R), then A is an ideal of R.*

***Proof*** **:**It follows that A is a left ideal of R, from the definition of a weakly standard ring. For any commutator $c \in C$ and $u, v \in R$, the identity (5) implies that

(c,uv) = u (c,v) + (c,u) v. (6)

Hence consider x((y,z),u).v and let c = (y,z). Then we get x(c,u).v = x.(c,u)v = x(c,uv) + xu.(c,v). Thus A is also a right ideal and the lemma is proved.

**Corollary:***The set B of all finite sums of elements of the form (C,R) R is an ideal of R.*

**Lemma 2 :***(C,R) = 0.*

***Proof :*** We assume that there exist elements d,e,f in R such that ((d,e),f) ≠ 0 and let Q = [q ∈ R ∣ q(C,R) = (C,R)q = 0 ]. It is easy to verify, using (6), that Q is an ideal of R. To prove that Q≠0 we suppose the contrary and let c = (x,y). From (5) we get that (zc,w) = z(c,w) + (z,w)c. So that z(c,w) is an element of $N_l$. Consequently, (c,w) (z,t,v) = 0 and also (z,t,v) (c,w) = 0. But this implies that all associators are in Q, R is associative, and this is a contradiction. We must therefore conclude that Q ≠ 0. However, AQ = QB = 0 implies that A = B = 0, and from this we are led to the fact that ((d,e),f) is an absolute zero divisor and hence zero. Hence the lemma is proved.

## Main results:

**Theorem 1 :***Let R be a prime ring. Then R is weakly standard if and only if R is either commutative or associative.*

***Proof :***We assume that R is a not associative weakly standard prime ring. It follows immediately that R does not have any non-zero absolute zero divisors, since the set of all such absolute zero divisors forms an ideal Z such that RZ = 0.

Next we assume that R is not commutative and consider a,b in R such that d = (a,b) ≠ 0. Let P = [p ∈ R ∣ pd = dp = 0], and observe that P is an ideal of R. We assume for the moment that P = 0. From (5), we see that (ab,b) = db. Thus 0 = ((ab,b),a) = $-d^2$ and this implies that d = 0, a contradiction. Therefore P ≠ 0. However, Rd is an ideal of R and Rdp = 0. The hypothesis on R forces the conclusion that Rd = 0, from which d is an absolute zero divisor and hence zero. This contradiction shows that R must be commutative. It remains only to remark that a commutative ring is flexible with (R,R) = 0, while an associative ring is automatically weakly standard.

We are now in a position to derive a structure theory for weakly standard rings. If R is simple and $R^2 = 0$, R is both associative and commutative. If R is simple and $R^2 \neq 0$, then R is prime. We consequently have

**Theorem 2 :***Let R be a simple ring. Then R is weakly standard if and only if R is either associative or commutative.*

Let us now recall that a ring R is called primitive if R contains a regular maximal right ideal E which contains no two-sided ideal of R other than the zero ideal. The following lemma was proved by Kleinfeld [2], using the hypothesis of accessibility. We do not use the hypothesis.

**Lemma 3:** *Let R be an arbitrary primitive ring. Then R is a prime ring.*

***Proof :*** Suppose E is a maximal right ideal of R such that ex-x is in E for all x in R and for some e in R. Let A and B be two ideals of R such that AB = 0 and assume that A ≠ 0. Then A⊄ E. So R = E + A and RB = EB ⊂ E. Hence eb∈ E and thus –b ∈ E, B ⊂ E and B = 0.

As an immediate application of the lemma we have

**Theorem 3 :**Let R be a primitive ring. Then R is weakly standard if and only if R is either associative or commutative.

Since it follows from a theorem of Brown [3] that a semi-simple ring is a subdirect sum of primitive rings, the structure theory for semi-simple weakly standard rings is completely determined by theorem 3.

Now we present an example of a weakly standard ring which is not accessible.

**Example 1:** Let A be an alternative ring of characterstic prime to 3 generated by three elements x,y,z such that (x,y,z) ≠ 0. Let B be the alternative ring obtained from A by setting all products in A containing at least four factors equal to zero, and by preserving all other sums and products. Then (x,y,z) ≠ 0 in B also. Moreover, B is flexible and trivially weakly standard. However, B does not satisfy (x,y,z) + (z,x,y) = 0, for other wise (x, y,z) = 0 as a consequence of the alternative law. This gives an example of a weakly standard ring that is not accessible.

Thus our results generalize and subsume those of Kleinfeld in [2].

**References:**

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