SOME FIXED POINT RESULTS IN METRIC AND CONE METRIC SPACES

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Abstract: In this paper, we prove some fixed point results for (r,s)-parameterized Suzuki type contraction mapping. Also, theorems in the setting of normal cone metric space are obtained.

Keywords: Fixed point, Complete metric space, Cone metric space, Suzuki type fixed point theorem.

Introduction: Following the important result in nonlinear functional analysis, Edelstein established a result and it states that every contractive mapping defined on a compact metric space has a unique fixed point. Among the generalized versions of Banach's result, Suzuki's result[1] defined a necessary and sufficient condition for completeness of a metric space in terms of existence of fixed point and many theorems has been made in this sequel[2- 4]. In 2009, Suzuki generalized the Edelstein's result by imposing a condition on contractive condition and it was a new approach to find fixed point. Suzuki's theorem[6] states that

Theorem 1.1 Let (X,d) be a compact metric space and let $f: X \to X$. Assume that

$1/2\ d(x,fx)<d(x,y)$ implies $d(fx,fy)<d(x,y)$

holds for all $x,y \in X$. Then f has a unique fixed point.

Following this trend, the researchers have been obtained many generalizations, see[9-12]. K -metric and K -normed spaces were introduced in the mid 20th century by using an ordered Banach space instead of the set of real numbers, as the co-domain for a metric. In [8], Huang and Zhang defined cone metric spaces by replacing the set of all real is denotes the interior of P (if $intP \neq \{0\}$, then the cone P is called solid). The cone P called is normal if there is a number $M > 0$such that for all $x,y \in E$, $0 \preccurlyeq x \preccurlyeq y$ implies $\|x\| \leq M\|y\|$, the least positive number satisfying this relation is called the normal constant of P. It is known that the previous condition is equivalent to the condition that there exists a norm 〚‖.‖〛_1 on E, equivalent to the given one, which is monotone, i.e., such that $0 \preccurlyeq x \preccurlyeq y$ implies that 〚‖x‖〛_1 ≤ 〚‖y‖〛_1.

Let X be a non-empty set. A function $d : X \times X \to E$ is called a cone metric if for all $x,y,z \in X$: (d_1) $0 \preccurlyeq d(x,y)$, (d_2) $d(x,y) = 0$ if and only if $x = y$, (d_3) $d(x,y) = d(y,x)$, (d_4) $d(x,y) \preccurlyeq d(x,z) + d(z,y)$. Then (X,d) is called an abstract(cone) metric space[8].

If the cone P is solid and normal, then using the previously mentioned equivalence, we can always suppose that the normal constant of P is M = 1, and that the given norm in E is monotone. In particular, we can take as a metric in X, the function D:X×X→ "R" given by $D(x,y) = \|d(x,y\|$, since it follows that this metric and cone metric give the same topologies on X , i.e., these spaces have the same collections of open, closed, bounded and compact sets, etc. In particular, in this case we can take that the following holds for all $x,y \in E$:

$0 \preccurlyeq x \ll y \Longrightarrow |(|x|)| < \|y\|$.

In this paper, we prove some fixed point theorems in which the contractivity of mapping is implied by a (r,s)-parameterized condition. Also, theorems in normal cone metric spaces are obtained. In addition, a theorem in the setting of compact cone metric space is obtained. Throughout this article, r and s are non-negative real numbers satisfying the condition (i) $s^2+2(s+r) \leq 1$ and $r \neq 0$.

hold. By the triangle inequality,

$sd(p,f^2 x_n) \leq sd(p,fx_n)+sd(fx_n,f^2 x_n)$

$sd(p,f^2 x_n) \leq sd(p,fx_n)+sd(x_n,fx_n)$

Now (5) becomes,

$d(p,fx_n)<rd(x_n,fx_n)+sd(p,fx_n)+sd(x_n,fx_n)$

$(1-s)d(p,fx_n)<(r+s)d(x_n,fx_n)$ $d(p,fx_n)<((r+s)/(1-s))d(x_n,fx_n)$ (6)

Now (4) becomes,

$d(x_n,p) \leq rd(x_n,fx_n)+sd(p,fx_n)$

$<rd(x_n,fx_n)+s((r+s)/(1-s))d(x_n,fx_n)$

$=((r(1-s)+s(r+s))/(1-s))d(x_n,fx_n)$

$=((s^2+r)/(1-s))d(x_n,fx_n)$ (7)

From (6) and (7), we have

$d(x_n,fx_n) \leq d(x_n,p)+d(p,fx_n)$

$<((s+r)/(1-s)+s^2/(1-s))d(x_n,fx_n)$

$=((s^2+s+2r)/(1-s))d(x_n,fx_n)$

$s^2+2(s+r) \leq 1$ implies $((s^2+s+2r)/(1-s)) \leq 1$,

then $d(x_n,fx_n)<((s^2+s+2r)/(1-s))d(x_n,fx_n)$

$\leq d(x_n,fx_n)$

which gives contradiction. Therefore, for any $n \in N$ either

$rd(x_n,fx_n)+sd(p,fx_n)<d(x_n,p)$ or (8)

$rd(fx_n,f^2 x_n)+sd(p,f^2 x_n)<d(fx_n,p)$(9)

Is true. This is equivalent to say that there are infinite subsets S_1 and S_2 on N such that $n \in S_1$, (8) holds and by our assumption, we have

$d(fx_n,fp) \leq \alpha d(x_n,p)+\beta d(x_n,fx_n)+$

$\gamma d(p,fp)+\delta d(x_n,fp)+\eta d(p,fx_n)$

As $n \to \infty$, we have

$d(p,fp) \leq (\gamma+\delta)d(p,fp)$

$(1-\gamma-\delta)d(p,fp) \leq 0$

$\Longrightarrow fp=p$

and $n \in S_2$, (9) holds, then

$d(ffx_n,fp) \leq \alpha d(fx_n,p)+\beta d(〚fx〛_n,ffx_n)$

$+\gamma d(p,fp)+\delta d(〚fx〛_n,fp)+\eta d(p,ffx_n)$

As $n \to \infty$, we have

$d(p,fp)=〚\lim〛_{(n\to\infty)} d(ffx_n,fp)$

$\leq 〚\lim〛_{(n\to\infty)} (\gamma d(fx_n,p)+\delta d(fx_n,fp))$

$=(\gamma+\delta)d(p,fp)$

$(1-\gamma-\delta)d(p,fp) \leq 0$

i.e., $fp=p$.

From both the cases, we have fp=p. Hence
$\leq(\rho^n+\rho^{(n+1)}+\cdots+\rho^{(m-1)})d(x_0,x_1)$
$\leq\rho^n/(1-\rho)\, d(x_0,x_1)$.
Since $\rho^n\longrightarrow 0$ as $n\longrightarrow\infty$, this implies $d(x_n,x_m)\longrightarrow 0$ as $m,n\longrightarrow\infty$. Therefore $\{x_n\}$ is Cauchy sequence in X. If X is complete, then the sequence $\{x_n\}$ is convergent to p(say) in X. Next we shall prove that p is the fixed point of f. If $fx_n\neq x_n, n\in N$, then
$rd(x_n,fx_n)+sd(fx_n,fx_n)<d(x_n,fx_n)$
$\Longrightarrow d(fx_n,ffx_n)<d(x_n,fx_n)$
Suppose that for $n\in N$,
$d(x_n,p)\leq rd(x_n,fx_n)+sd(p,fx_n)$ and
$d(p,fx_n)\leq rd(fx_n,f^2 x_n)+sd(p,f^2 x_n)$
i.e.,$(p,fx_n)<rd(x_n,fx_n)+sd(p,f^2 x_n)$
hold. Using the same arguments in Theorem 3.1, we have that there are infinite subsets S_1 and S_2 of N such that $n\in S_1$, $rd(x_n,fx_n)+sd(p,fx_n)<d(x_n,p)$ holds and by our assumption, we have
$d(fx_n,fp)\leq\alpha d(p,fp)[1+d(x_n,fx_n)+d(p,fx_n)]/(1+d(x_n,p))+\beta d(x_n,fx_n)$
As $n\longrightarrow\infty$, we have
$d(p,fp)\leq\alpha d(p,fp)$
$(1-\alpha)d(p,fp)\leq 0$
$\Longrightarrow fp=p$
and $n\in S_2$, $rd(fx_n,f^2 x_n)+sd(p,f^2 x_n)<d(fx_n,p)$ holds, then
$d(ffx_n,fp)\leq(\alpha d(p,fp)[1+d(fx_n,ffx_n)+d(p,ffx_n)])/(1+d(fx_n,p))+\beta d(p,fx_n)$
As $n\longrightarrow\infty$, we have that fp=p. From both the cases, we have fp=p. Hence we have shown that p is the fixed point of f. To prove uniqueness, suppose there exists another fixed point q(say) of f. By (10), we have
$rd(p,fp)+sd(q,fp)<d(p,q)$ implies
$d(fp,fq)\leq\alpha d(q,fq)[1+d(p,fp)+d(q,fp)]/(1+d(p,q))+\beta d(p,q)$
i.e., $d(p,q)\leq\beta d(p,q)$
which imply that p=q. Hence f has a unique fixed point, i.e., fp=p.
Corollary Let f be a self-mapping of a
$d(Tx,Ty)\preccurlyeq(\alpha d(y,Ty)[1+d(x,Tx)+d(y,Tx)])/(1+d(x,y))+\beta d(x,y)$ (13)
where α and β are in [0,1) such that $\alpha+\beta<1$, then T has a unique fixed point.
Corollary Let (X,d) be acomplete cone metric space, P normal with solid and $T:X\longrightarrow X$ be a mapping on X. If T satisfies the condition that
$rd(x,fx)+sd(y,fx)-d(x,y)\notin intP$
$\Longrightarrow d(Tx,Ty)\preccurlyeq\alpha d(x,Tx)+\beta d(y,Ty)$
where $\alpha,\beta\geq 0$ and $\alpha\neq 0$ such that $\alpha+\beta<1$, then T has a unique fixed point.
Corollary 3.4 Let (X,d) be a complete cone metric space, P normal with solid and $T:X\longrightarrow X$ be a mapping on X. If X satisfies the condition that
$rd(x,fx)+sd(y,fx)-d(x,y)\notin intP$
$\Longrightarrow d(Tx,Ty)\preccurlyeq 1/2\, d(x,Ty)+1/2\, d(y,Tx)$
then T has a unique fixed point.
Theorem 3.5 Let (X,d) be a compact cone metric space, P be a normal and solid cone and let $T:X\rightarrow X$. If for all $x,y\in X$ with $x\neq y$,
$rd(x,fx)+sd(y,fx)-d(x,y)\notin intP$
$\Longrightarrow d(Tx,Ty)\ll d(x,y)$ (14)
holds, then T has a unique fixed point in X.
Proof Since the cone P is normal, there exists a norm $⟦\|.\|⟧_1$ which is equivalent to the norm $\|.\|$ defined on E for which the normal constant of P is 1 and the norm $⟦\|.\|⟧_1$ on E is monotone, i.e., $\theta\preccurlyeq x\ll y \Longrightarrow |(|x|)|_1<|(|y|)|_1$. Define $D(x,y)=|(|d(x,y)|)|_1$
Then D is a (real valued) metric and the space (X,D) is compact. Let us prove that the mapping T on X satisfies the conditionr$D(x,fx)+sD(y,fx)<D(x,y)$
$\Longrightarrow D(Tx,Ty)<D(x,y)$ (15)
Now, we have that if $rD(x,Tx)+sD(y,Tx)<D(x,y)$ holds for $x,y\in X$, then $rd(x,Tx)+sd(y,Tx)-d(x,y)\notin intP$. For, to the contrary, rd(x,Tx)+
numbers with the ordered Banach space and generalized the fundamental result in fixed point theory, Banach Contraction Mapping Principle. Also they described the notion of Cauchy sequences and convergent sequences via interior of the cone by which the order is defined. Banach contraction principle in the setting of normal cone metric spaces follows as
Theorem 1.2 Let (X,d) be a complete cone metric space over a normal solid cone. Suppose that a mapping T∶ X → X satisfies the contractive condition
$d(T x,T y)\preccurlyeq\lambda d(x,y)$ (1)
for all $x,y\in X$, where $\lambda\in[0,1)$ is a constant. Then T has a unique fixed point in X and for any $x\in X$, the iterative sequence {T n x} converges to the fixed point.
After this work, many authors proved fixed point re- sults in normal cone metric spaces, which generalize the notion of metric space.
2. Preliminaries: Let E be a real Banach space. A subset P of E is called a cone if: (i) P is closed, non-empty and P = {0}, (ii) $ax + by\in P$, for $x,y\in P$ and non- negative real numbers a,b (iii) $P\cap(-P) = \{0\}$. For a given cone $P\subseteq E$, we can define a partial order order $\preccurlyeq$ with respect to P by $x\preccurlyeq y$ if and only if $y - x\in P$.$x\prec y$ will stand for $x\preccurlyeq y$ and $x = y$, while $x\ll y$ will stand for $y - x\in intP$, where intP
3 Main results
Our first theorem generalize the Suzuki's result with (r,s)-parametrization and it follows
Theorem 2.1 Let (X,d) be a complete metric space. If a self-mapping $f:X\rightarrow X$ satisfies $rd(x,f x) + sd(y,f x) <d(x,y)$ Implies $d(f x,f y)\leq$
$\alpha d(x, y) + \beta d(x, f x) +\gamma d(y, f y) +\delta d(x, f y) + \eta d(y, f x)$, (2)
where $r,s,\alpha,\beta,\gamma,\delta,\eta\in R$ such that $\alpha+\beta+\gamma+2\delta<1$ with $\alpha+\gamma+\beta>0$, then f has a fixed point. If $\eta<\beta+\gamma+\delta$, then the fixed point is unique.
Proof: Let $x_0\in X$ be arbitrary. If $x_0 = x_0$, there is nothing to prove. Consider the sequence of iterations $x_1= f x_0, x_2=fx_1,\ldots,x_n= f x_{(n-1)}$, in X. Assume that $f x_n\neq x_n$ for all $n\in N$. By (2), we have
$rd(x_n,fx_n)+sd(fx_n,fx_n)<d(x_n,fx_n)$
$\Longrightarrow d(fx_n,ffx_n)\leq\alpha d(x_n,fx_n)+\beta d(x_n,fx_n)+\gamma d(fx_n,ffx_n)+\delta d(x_n,ffx_n)+\eta d(fx_n,fx_n)$
Then $d(fx_n,f^2 x_n)\leq((\alpha+\beta+\delta)/(1-\gamma-\delta))d(x_n,fx_n)$. Let

$\rho=(\alpha+\beta+\delta)/(1-\gamma-\delta)$. Since $\alpha+\beta+\gamma+2\delta<1$, then $\rho<1$. Let $\varepsilon>0$ be any number. For $m\geq n$, we have

$d(x_n,x_m)\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\dots+d(x_{m-1},x_m)$

$\leq(\rho^n+\rho^{n+1}+\cdots+\rho^{m-1})d(x_0,x_1)$

$\leq\rho^n/(1-\rho)\, d(x_0,x_1)$

Since $\rho^n\to 0$ as $n\to\infty$, the term, for some $n_0\in N$, $d(x_n,x_m)<\varepsilon$, for all $m\geq n\geq n_0$. Therefore, the sequence $\{x_n\}$ is Cauchy sequence in X and completeness of X implies the existence of an element p in X such that $\{x_n\}$ converges to p. Next we shall prove that p is the fixed point of f. If $fx_n\neq x_n, n\in N$, then

$rd(x_n,fx_n)+sd(fx_n,fx_n)<d(x_n,fx_n)$

$\Rightarrow d(fx_n,ffx_n)<d(x_n,fx_n)$ (3)

Suppose that for $n\in N$,

$d(x_n,p)\leq rd(x_n,fx_n)+sd(p,fx_n)$ and (4)

$d(p,fx_n)\leq rd(fx_n,f^2x_n)+sd(p,f^2x_n)$ $(p,fx_n)<rd(x_n,fx_n)+sd(p,f^2x_n)$ (5)

we have shown that p is the fixed point of f.

To prove uniqueness, suppose there exists another fixed point q(say) of f.By hypothesis, we have

$rd(p,fp)+sd(q,fp)<d(p,q)$

$\Rightarrow d(fp,fq)\leq\alpha d(p,q)+\beta d(p,fp)+\gamma d(q,fq)+\delta d(p,fq)+\eta d(p,fp)$

$=(\alpha+\delta+\eta)d(p,q)$

Since $\eta<\beta+\gamma+\delta$, then $\alpha+\delta+\eta<\alpha+\beta+\gamma+2\delta\leq\alpha+\beta+2(\gamma+\delta)\leq 1$ and we have $d(fp,fq)<d(p,q)$ which contradicts. Hence f has a unique fixed point, i.e., fp=p.

Corollary 3.1 Let (X,d) be a complete ,etroc s[ace amd $f:X\longrightarrow X$ be a mapping. If the mapping f on X satisfies

$rd(x,fx)+sd(y,fx)<d(x,y)$

$\Rightarrow d(fx,fy)\leq\alpha d(x,y)$,

where $\alpha\in(0,1)$, then f has a unique fixed point.

Proof :If we have $\beta=\gamma=\delta=\eta=0$ and $\alpha\neq 0$ in Theorem 3.1, we can get the result.

Theorem 3.2 Let (X,d) be a metric space and $fX\longrightarrow X$ be a mapping on X. If f satisfies the condition that

$rd(x,fx)+sd(y,fx)<d(x,y)$

implies $d(fx,fy)\leq$

$(\alpha d(y,fy)[1+d(x,fx)+d(y,fx)])/(1+d(x,y))+\beta d(x,y)$ (10)

where α and β are in (0,1) such that $\alpha+\beta<1$, then for any $x_0\in X$, the sequence $x_n=fx_{n-1}$ is Cauchy sequence. If the space X is complete, the f has a unique fixed point.

Proof Let x_0 be arbitrary in X. Consider the sequence of terms $x_n=fx_{n-1}$ for n=1,2,3,…. If $x_n\neq fx_n$, then $rd(x_n,fx_n)+sd(fx_n,fx_n)<d(x_n,fx_n)$ and by assumption,

$d(fx_n,ffx_n)\leq$

$(\alpha d(fx_n,ffx_n)[1+d(x_n,fx_n)+d(fx_n,fx_n)])/(1+d(x_n,fx_n))+\beta d(x_n,fx_n)$

$=(\alpha d(fx_n,ffx_n)[1+d(x_n,fx_n)])/(1+d(x_n,fx_n))++\beta d(x_n,fx_n)$

$\Rightarrow d(x_{n+2},x_{n+1})\leq\rho d(x_{n+1},x_n)$

where $\rho=\beta/(1-\alpha)<1$ and by triangle inequality, we have

$d(x_n,x_m)\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{m-1},x_m)$

complete metric space such that $rd(x,fx)+sd(y,fx)<d(x,y)$ imply

$d(fx,fy)\leq\alpha d(y,fy)[1+d(x,fx)]/(1+d(x,y))+\beta d(x,y)$

where α and β are in (0,1) such that $\alpha+\beta<1$, then f has a unique fixed point.

Proof The proof of this Corollary comes from the arguments of Theorem 3.2 easily.

Remark 3.1:If we take $\alpha=0$ in Theorem 3.2, we have the **Corollary 3.1:**Next we prove theorems in complete and compact cone metric spaces.

Theorem 3.3:Let (X,d) be a complete cone metric space, P be a normal and solid cone P and let $T:X\longrightarrow X$. If for all $x,y\in X$ with $x\neq y$

$rd(x,fx)+sd(y,fx)-d(x,y)\notin intP$

$\Rightarrow d(fx,fy)\preccurlyeq\alpha d(x,y)+\beta d)$

$\gamma d(y,fy)+\delta d(x,fy)+\eta d(y,fx)$ (11)

where $\alpha,\beta,\gamma,\delta,\eta\in R_+$ such that $\alpha+\beta+\gamma+2\delta<1$, then f has a fixed point. If $\eta<\beta+\gamma+\delta$, then the fixed point is unique.

Proof Since the cone P is normal, without loss of generality, we can suppose that the normal constant of P is 1 and the given norm in E is monotone. Now define $D(x,y)=|(|d(x,y)|)|$. Then D is a real valued metric and the space (X,D) is (together with (X,d)) compact. Next we shall prove that T satisfies the condition

$rD(x,fy)+sD(y,fx)<D(x,y)$

$\Rightarrow D(fx,fy)\leq\alpha D(x,y)+\beta D(x,fx)+\delta D(x,fy)+\eta D(y,fx)$, (12)

For $x,y\in X$, if $rD(x,fx)+sD(y,fx)<D(x,y)$, then $rd(x,fx)+sd(y,fx)-d(x,y)\notin intP$, for this, suppose that $rd(x,fx)+sd(y,fx)-d(x,y)\in intP$implies

$d(x,y)\leq rd(x,fx)+sd(y,fx)$ and $D(x,y)\leq rD(x,fx)+D(y,fx)$, by monotonicity of the norm, contradicts. Now (11) implies the inequality (12). Thus D satisfies the required condition and the proof follows.

Theorem 3.4 Let (X,d) be a complete cone metric space, P normal with solid and $T:X\longrightarrow X$ be a mapping on X. If T satisfies the condition that

$rd(x,fx)+sd(y,fx)-d(x,y)\notin intP$

$sd(y,Tx)-d(x,y)\in intP$, i.e., $d(x,y)\ll rd(x,Tx)+sd(y,Tx)$. It follows that $D(x,y)<rD(x,Tx)+sD(y,Tx)$,

contradiction with the assumption. Therefore, $x,y\in X$, (15) holds. Now the result follows Theorem 2.1 in [12].

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