OSCILLATION OF GENERALIZED SECOND ORDER SUBLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS

V.CHANDRASEKAR, A.BENEVATHO JAISON

Abstract: In this paper, the authors present new oscillation criteria for the generalized second order sublinear neutral delay difference equation

$$\Delta_\ell(a(k)\Delta_\ell(u(k)+p(k)u(k-\tau\ell)))+q(k)u^\gamma(k-\sigma\ell)=0, \quad (1)$$

where $k\in[0,\infty), 0<\gamma<1$ is a quotient of odd positive integers, τ and σ are fixed nonnegative integers, $p(k)$ and $q(k)$ are real valued functions and $a(k)>0$ such that

$$\sum_{k=k_0}^{\infty}\frac{1}{a(k)}=\infty, 0\le p(k)<1 \text{ for all } k\ge 0 \text{ and } q(k)\ge 0. \quad (2)$$

Keywords: Oscillatory Behaviour, Sublinear, Neutral, Delay.

Introduction: Difference equations usually described the evaluation of some certain phenomena over time and are also important in describing dynamically discrete system, see [2]. For example, in the numerical integration, the standard approach is to use the difference equations. Similarly, the population dynamics have discrete generations; the size of the $(k+1)^{th}$ generation $u(k+1)$ is a function of the k^{th} generation $u(k)$. This can be expressed as difference equation is of the form

$$u(k+1)=f(u(k)),$$

see for example [1]. Further, the concept of difference equations with many examples in applications such as asymptotic behavior of solutions of difference equations were studied extensively by Elaydi [8] where the analytic and geometric approaches were also combined in order to studying difference equations. Further, in [2], both classical and modern treatments of the difference equations were presented in excellent form. For related results on difference equations, see [1], [8].

Recently, there has been increasing interest in the study of oscillatory behaviors to the solutions of linear and nonlinear difference equations. The reference [9], [12], [14], concern with the oscillation of non-linear difference equations. The reference [4]-[6], [13], [15] studied respectively the oscillation of quasilinear, sublinear, half linear, and super linear difference equations.

To the best of our knowledge, nothing is known regarding the qualitative behavior of solutions of equation (1) in the sublinear case. Hence our aim of this paper is to give several oscillation criteria of Equation (1) when (2) holds.

In this paper, we use the following assumptions:

By a solution of (1) we mean a nontrivial function $u(k)$ which is defined for $k\ge -K,$ where $K=\max\{\tau,\sigma\},$ and satisfies Equation (1) for $k\in[0,\infty).$ Clearly if

$$u(k)=A(k) \text{ for } k\in[-K,0] \quad (3)$$

are given, then equation (1) has a unique solution satisfying the initial conditions (3).

Preliminaries:

Before stating and proving our results, we present some notation.

(i) $j=k-\left[\frac{k}{\ell}\right]\ell,$ $\left[\frac{k}{\ell}\right]$ means integer part of $\frac{k}{\ell}$

(ii) $N(1)=\{1,2,3,\cdots\},$

(iii) $N_\ell(j)=\{j,j+\ell,j+2\ell,\cdots\}.$

In this section, we present some basic definitions and preliminary results which will be used in the subsequent discussions.

Definition 2.1 [7] For a real valued function $u(k)$, the generalized difference operator Δ_ℓ and its inverse are respectively, defined as

$\Delta_\ell u(k)=u(k+\ell)-u(k), k\in[0,\infty), \ell\in(0,\infty),$ and if

$\Delta_\ell v(k)=u(k)$ *then* $v(k)=\Delta_\ell^{-1}u(k)+c_j,$

where c_j's are constant for all $k\in N_\ell(j)$.

Definition 2.2 [7] For $\lambda\in N(1),$ the generalized polynomial factorial is defined by

$$k_\ell^{(\lambda)}=k(k-\ell)(k-2\ell)\cdots(k-(\lambda-1)\ell). \quad (4)$$

Lemma 2.3 [7] Let $\ell\in[0,\infty).$ Then

$$\Delta_\ell k_\ell^{(\lambda)}=(\lambda\ell)k_\ell^{(\lambda-1)}.$$

Definition 2.4 A solution $u(k)$ of (1) is said to be oscillatory if for every $k_1>0$ there exists a real $k\ge k_1$ such that $u(k)u(k+\ell)\le 0.$ Otherwise it is nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Definition 2.5 [5] A function $f(n,y_1,y_2,\cdots,y_m)$ is said to be strongly sublinear if there exists constant $\beta\in(0,1),$ β is a quotient of odd positive integers and

$d > 0$ such that $|y|^{-\beta}|f(n, y, y, \cdots, y)|$ is nonincreasing in $|y|$ for $0 < |y| \le d$.

Lemma 2.6 [7] (Product Formula)

Let $u(k)$ and $v(k)$ be any two real valued functions. Then

$$\Delta_\ell\{u(k)v(k)\} = u(k+\ell)\Delta_\ell v(k) + v(k)\Delta_\ell u(k) = v(k+\ell)\Delta_\ell u(k) + u(k)\Delta_\ell v(k).$$

Definition 2.7 [7] Let $u(k)$ be real a valued function. Then for $k \in [0, \infty)$,

$$\Delta_\ell^{-1} u(k)|_j^k = \sum_{r=1}^{\left[\frac{k}{\ell}\right]} u(k - r\ell).$$

Main results: Throughout this paper, we will assume that $\Delta_\ell a(k) \ge 0$ and k by $k\ell + j$.

Theorem 3.1 Assume that (2) holds and there exists a positive real valued function $\rho(k)$ such that for every $\alpha \ge \ell$,

$$\limsup_{k\to\infty} \sum_{r=0}^{k} \Big[\rho(r)Q(r) - \frac{a(r-\sigma\ell)}{4\gamma\rho(r)} (\alpha(r+\ell(1-\sigma)))^{1-\gamma}(\Delta_\ell \rho(r))^2\Big] = \infty, \quad (5)$$

where $Q(k) = q(k)(1 - p(k-\sigma\ell))^\gamma$. Then every solution of equation (1) oscillates.

Proof. Assume for the sake of contradiction that equation (1) has a nonoscillatory solution $u(k)$, we may assume without loss of generality that $u(k-K) > 0$ for $k \ge k_0 > 0$ (the case when $u(k) < 0$ is similar and hence is omitted).

Set $z(k) = u(k) + p(k)u(k-\tau\ell)$. (6)

By, assumption (2) we have $z(k) > 0$ for $k \ge k_0$ and from (1) it follows that for

$$\Delta_\ell(a(k)\Delta_\ell z(k)) = -q(k)u^\gamma(k-\sigma\ell) \le 0,$$

and so $a(k)\Delta_\ell z(k)$ is an eventually nonincreasing real valued function. We first show that $\Delta_\ell z(k) \ge 0$ for $k \ge k_0$. In fact, if there exists a real $k_1 \ge k_0$ such that $a(k_1)\Delta_\ell z(k_1) = c < 0$, then $a(k)\Delta_\ell z(k) \le c$ for $k \ge k_1$, that is

$$\Delta_\ell z(k) \le \frac{c}{a(k)} \text{ and, hence}$$

$$z(k) \le z(k_1) + c\sum_{r=k_1}^{k-1} \frac{1}{a(r)} \to -\infty \;\; as \;\; k \to \infty,$$

which contradicts the fact that $z(k) > 0$ for $k \ge k_0$. Also we claim that $\Delta_\ell^2 z(k) \le 0$. If not there exists $k_1 \ge k_0$ such that $\Delta_\ell^2 z(k) > 0$ for $k \ge k_1$ and this implies that $\Delta_\ell z(k+\ell) > \Delta_\ell z(k)$, so that since $\Delta_\ell a(k) \ge 0$, and

$$a(k+\ell)\Delta_\ell z(k+\ell) > a(k+\ell)\Delta_\ell z(k) \ge a(k)\Delta_\ell z(k)$$

this contradicts the fact that $a(k)\Delta_\ell z(k)$ is nonincreasing real valued function, then $\Delta_\ell^2 z(k) \le 0$ and therefore we have

$$z(k) > 0, \Delta_\ell z(k) \ge 0, \Delta_\ell^2 z(k) \le 0 \text{ for } k \ge k_0, \quad (7)$$

and then from (6) and (7), we have $u(k) \ge (1-p(k))z(k)$ and this implies that for $k \ge k_1 = k_0 + \sigma\ell$

$$u(k-\sigma\ell) \ge (1-p(k-\sigma\ell))z(k-\sigma\ell).$$

From equation (1) and the last inequality, we have for $k \ge k_1$,

$$\Delta_\ell(a(k)\Delta_\ell z(k)) + Q(k)z^\gamma(k-\sigma\ell) \le 0. \quad (8)$$

Define the function

$$w(k) = \rho(k)\frac{a(k)\Delta_\ell z(k)}{z^\gamma(k-\sigma\ell)}. \quad (9)$$

Then $w(k) > 0$ and

$$\Delta_\ell w(k) = a(k+\ell)\Delta_\ell z(k+\ell)\Delta_\ell\left[\frac{\rho(k)}{z^\gamma(k-\sigma\ell)}\right] + \frac{\rho(k)\Delta_\ell(a(k)\Delta_\ell z(k))}{z^\gamma(k-\sigma\ell)}. \quad (10)$$

From the fact that $\Delta_\ell z(k) \ge 0$ and $\Delta_\ell(a(k)\Delta_\ell z(k)) \le 0$, for $k \ge k_1$,

wehave

$a(k-\sigma\ell)\Delta_\ell z(k-\sigma\ell) \ge a(k+\ell)\Delta_\ell z(k+\ell)$ and

$$z(k+(1-\sigma)\ell) \ge z(k-\sigma\ell). \quad (11)$$

Then, from equation (1), (9) and (10), we get

$$\Delta_\ell w(k) = -\rho(k)Q(k) + \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)} w(k+\ell) - \frac{\rho(k)a(k+\ell)\Delta_\ell z(k+\ell)\Delta_\ell z^\gamma(k-\sigma\ell)}{z^\gamma(k+\ell(1-\sigma))z^\gamma(k-\sigma\ell)}. \quad (12)$$

From (11) and (12), we have

$$\Delta_\ell w(k) \le -\rho(k)Q(k) + \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)} w(k+\ell)$$

$$-\frac{\rho(k)a(k+\ell)\Delta_\ell z(k+\ell)\Delta_\ell z^\gamma(k-\sigma\ell)}{(z^\gamma(k+\ell(1-\sigma)))^2}. \quad (13)$$

Now, by using the inequality(cf.[3],p.39) $(u-v)\gamma \geq \gamma u^{\gamma-1}(u-v)$ for all $u \geq v > 0$ and $0<\gamma\leq 1,$ we find that

$$\Delta_\ell z(k-\sigma\ell)\gamma = z(k+\ell(1-\sigma))\gamma - z(k-\sigma\ell)\gamma$$
$$\geq \gamma(z^{\gamma-1}(k+\ell(1-\sigma)))$$
$$(z(k+\ell(1-\sigma))-z(k-\sigma\ell))$$
$$=\gamma(z^{\gamma-1}(k+\ell(1-\sigma)))\Delta_\ell z(k-\sigma\ell). \quad (14)$$

Substitute from (14) in (13),

we have $\Delta_\ell w(k) = -\rho(k)Q(k) + \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)} w(k+\ell)$

$$-\gamma(z^{\gamma-1}(k+\ell(1-\sigma)))\rho(k)a(k+\ell)$$
$$\times\left(\frac{\Delta_\ell z(k+\ell)\Delta_\ell z(k-\sigma\ell)}{(z^\gamma(k+\ell(1-\sigma)))^2}\right). \quad (15)$$

Again, from (11) in (15), we obtain

$$\Delta_\ell w(k) = -\rho(k)Q(k) + \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)} w(k+\ell)$$
$$-\gamma(z^{\gamma-1}(k+\ell(1-\sigma)))\rho(k)$$
$$\times\left(\frac{(a(k+\ell))^2}{a(k-\sigma\ell)}\frac{(\Delta_\ell z(k+\ell))^2}{(z^\gamma(k+\ell(1-\sigma)))^2}\right), \quad (16)$$

and hence,

$$\Delta_\ell w(k) \leq -\rho(k)Q(k) + \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)} w(k+\ell)$$
$$-(z^\gamma(k+\ell(1-\sigma)))^2$$
$$\left(\frac{\gamma\rho(k)(a(k+\ell))^2(\Delta_\ell z(k+\ell))^2}{a(k-\sigma\ell)(z^{1-\gamma}(k+\ell(1-\sigma)))}\right) \quad (17)$$

From (9) and (17), we find

$$\Delta_\ell w(k) = -\rho(k)Q(k) + \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)} w(k+\ell) -$$
$$\frac{\gamma\rho(k)w^2(k+\ell)}{a(k-\sigma\ell)(\rho(k+\ell))^2(z^{1-\gamma}(k+\ell(1-\sigma)))}. \quad (18)$$

From (7), we conclude that

$$z(k) \leq z(k_0) + \Delta_\ell z(k_0)(k-k_0), \quad k \geq k_0,$$

and consequently there exists a $k_1 \geq k_0$ and appropriate constant $\alpha \geq \ell$ such that

$$z(k) \leq \alpha k \quad for\ k \geq k_1,$$

and this implies that

$$z(k+\ell(1-\sigma)) \leq \alpha(k+\ell(1-\sigma))$$

for $k \geq k_2 = k_1 + (\sigma-1)\ell,$ and, hence

$$\frac{1}{(z^{1-\gamma}(k+\ell(1-\sigma)))} \geq \frac{1}{(\alpha(k+\ell(1-\sigma)))^{1-\gamma}}.$$

Substitute from the last inequality in (18), we find

$$\Delta_\ell w(k) = -\rho(k)Q(k) + \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)} w(k+\ell)$$
$$-\frac{\gamma\rho(k)w^2(k+\ell)}{(\rho(k+\ell))^2 a(k-\sigma\ell)(\alpha(k+\ell(1-\sigma)))^{1-\gamma}}$$
$$=-\rho(k)Q(k)$$
$$+a(k-\sigma\ell)(\alpha(k+\ell(1-\sigma)))^{1-\gamma}\frac{(\Delta_\ell\rho(k))^2}{4\gamma\rho(k)}$$
$$-\left[\frac{\sqrt{\gamma\rho(k)}w(k+\ell)}{\rho(k+\ell)\sqrt{(\alpha(k+\ell(1-\sigma)))^{1-\gamma}a(k-\sigma\ell)}}\right.$$
$$\left.-\frac{\sqrt{(\alpha(k+\ell(1-\sigma)))^{1-\gamma}a(k-\sigma\ell)}\Delta_\ell\rho(k)}{2\sqrt{\gamma\rho(k)}}\right]^2$$
$$< -\left[\rho(k)Q(k) - (\alpha(k+\ell(1-\sigma)))^{1-\gamma}\right.$$
$$\left.\times\frac{a(k-\sigma\ell)(\Delta_\ell\rho(k))^2}{4\gamma\rho(k)}\right]. \quad (19)$$

Then, we have

$$\Delta_\ell w(k) < -\left[\rho(k)Q(k) - a(k-\sigma\ell)\right.$$
$$\left.\frac{(\alpha(k+\ell(1-\sigma)))^{1-\gamma}(\Delta_\ell\rho(k))^2}{4\gamma\rho(k)}\right]. \quad (20)$$

Summing (20) from k_2 to k , we obtain

$$-w(k_2) < w(k+\ell) - w(k_2)$$
$$< -\sum_{r=k_2}^{k}\left[\rho(r)Q(r)\right.$$
$$\left.-\frac{a(r-\sigma\ell)(\alpha(r+\ell(1-\sigma)))^{1-\gamma}(\Delta_\ell\rho(r))^2}{4\gamma\rho(r)}\right],$$

which yields

$$\sum_{r=k_2}^{k}\left[\rho(r)Q(r) -\right.$$
$$\left.\frac{a(r-\sigma\ell)(\alpha(r+\ell(1-\sigma)))^{1-\gamma}(\Delta_\ell\rho(r))^2}{4\gamma\rho(r)}\right] < c_1,$$

for all large k , which is contrary to (5). The proof is complete.

Remark 3.2 Note that from Theorem 3.1, we can obtain different conditions for oscillation of all solutions of equation (1) when (2) holds by different choices of $\rho(k).$ Let $\rho(k) = k^\lambda, k \geq k_0$ and $\lambda > 1$ is a constant. By Theorem 3.5 we have the following result.

Corollary 3.3 Assume that all the assumption of Theorem 3.1 hold, except the condition (5) is replaced by

$$\limsup_{k\to\infty}\sum_{s=k_0}^{k}\Big[s^{\lambda}Q(s)-a(s-\sigma\ell)\frac{(\alpha(s+\ell(1-\sigma)))^{1-\gamma}(\Delta_\ell s^{\lambda})^2}{4\gamma s^{\lambda}}\Big]=\infty, \quad (21)$$

where $Q(k)=q(k)(1-p(k-\sigma\ell))^{\gamma}$. Then every solution of equation (1) oscillates.

Remark 3.4 When $\gamma=1$, equation (1) reduces to the linear delay generalized difference equation

$$\Delta_\ell(a(k)(\Delta_\ell u(k))+p(k)u(k-\tau\ell))+q(k)u(k-\sigma\ell)=0, k\in[0,\infty) \quad (22)$$

and the condition (5) in Theorem 3.1 reduces to

$$\limsup_{k\to\infty}\sum_{r=0}^{k}\Big[\rho(r)q(r)(1-p(r-\sigma\ell))-\frac{a(r-\sigma\ell)(\Delta_\ell\rho(r))^2}{4\rho(r)}\Big]=\infty. \quad (23)$$

Then Theorem 3.1 and Theorem 1 in [10] are the same in linear case. Also when $p(k)=0$ and $\gamma=1$ Theorem 3.1 and Corollary 1 in [11] are the same.

Theorem 3.5 Assume that (2) holds, and let $u(k)$ be a real valued function on $[0,\infty)$. Also, we assume that there exists a two real valued functions $\{H(t,k): t\ge k\ge 0\}$ such that

i) $H(t,t)=0$ for $t\ge 0$, (ii) $H(t,k)>0$ for $t>k>0$ (iii) $\Delta_{\ell(k)}H(t,k)=H(t,k+\ell)-H(t,k)$.

If

$$\limsup_{t\to\infty}\frac{1}{H(t,0)}\sum_{k=k_0}^{t-1}\Big[H(t,k)\rho(k)Q(k)-\frac{\rho^2(k+\ell)}{4\overline{\rho(k)}}\Big(h(t,k)\sqrt{H(t,k)}-\frac{\Delta_\ell\rho(k)}{\rho(k+\ell)}H(t,k)\Big)^2\Big]=\infty, \quad (24)$$

where

$$h(t,k)=\frac{-\Delta_{\ell(k)}H(t,k)}{\sqrt{H(t,k)}},$$

$$\overline{\rho(k)}=\frac{\gamma\rho(k)}{(\alpha(k+\ell(1-\sigma)))^{1-\gamma}a(k-\sigma\ell)},$$

then every solution of equation (1) oscillates.

Proof. Proceeding as in Theorem 3.1, we assume that equation (1) has a nonoscillatory solution, say $u(k)>0$ and $u(k-\sigma\ell)>0$ for all $k\ge k_0$. From the proof of Theorem 3.1 we obtain (19) for all $k\ge k_2$ From (19), since $\Delta_\ell\rho(k)\le 0$, we have for $k\ge k_2$

$$\Delta_\ell w(k)\le-\rho(k)Q(k)+\frac{\Delta_\ell\rho(k)w(k+\ell)}{\rho(k+\ell)}-\frac{\overline{\rho(k)}w^2(k+\ell)}{(\rho(k+\ell))^2}, \quad (25)$$

or

$$\rho(k)Q(k)\le-\Delta_\ell w(k)+\frac{\Delta_\ell\rho(k)w(k+\ell)}{\rho(k+\ell)}-\frac{\overline{\rho(k)}w^2(k+\ell)}{(\rho(k+\ell))^2}. \quad (26)$$

Therefore, we have

$$\sum_{k=k_2}^{t-1}H(t,k)\rho(k)Q(k)\le-\sum_{k=k_2}^{t-1}H(t,k)\Delta_\ell w(k)+\sum_{k=k_2}^{t-1}\frac{H(t,k)\Delta_\ell\rho(k)w(k+\ell)}{\rho(k+\ell)}-\sum_{k=k_2}^{t-1}\frac{H(t,k)\overline{\rho(k)}}{(\rho(k+\ell))^2}w^2(k+\ell), \quad (27)$$

which yields, after summing by parts,

$$\sum_{k=k_2}^{t-1}H(t,k)\rho(k)q(k+\ell)\le H(t,k_2)w(k_2)+\sum_{k=k_2}^{t-1}w(k+\ell)\Delta_{\ell(k)}H(t,k)+\sum_{k=k_2}^{t-1}H(t,k)\frac{\Delta_\ell\rho(k)}{\rho(k+\ell)}w(k+\ell)-\sum_{k=k_2}^{t-1}H(t,k)\frac{\overline{\rho(k)}}{(\rho(k+\ell))^2}w^2(k+\ell)$$

$$=H(t,k_2)w(k_2)-\sum_{k=k_2}^{t-1}h(t,k)\sqrt{H(t,k)}w(k+\ell)+\sum_{k=k_2}^{t-1}\frac{\Delta_\ell\rho(k)}{\rho(k+\ell)}\big(H(t,k)w(k+\ell)\big)-\sum_{k=k_2}^{t-1}\frac{H(t,k)\overline{\rho(k)}}{(\rho(k+\ell))^2}w^2(k+\ell)=H(t,k_2)w(k_2)+$$

$$\sum_{k=k_2}^{t-1}\frac{(\rho(k+\ell))^2}{4\overline{\rho(k)}}\Big(h(t,k)-\frac{\Delta_\ell\rho(k)}{\rho(k+\ell)}\sqrt{H(t,k)}\Big)^2$$

$$-\sum_{k=k_2}^{t-1}\left[\frac{\sqrt{H(t,k)\overline{\rho(k)}}}{\rho(k+\ell)}w(k+\ell)+\frac{\rho(k+\ell)}{2\sqrt{H(t,k)}}\right.$$

$$\left.\frac{1}{\sqrt{\overline{\rho(k)}}}\left(h(t,k)\sqrt{H(t,k)}-\frac{\Delta_\ell\rho(k)}{\rho(k+\ell)}H(t,k)\right)\right]^2.$$

Then, $\sum_{k=k_2}^{t-1}\left[H(t,k)\rho(k)q(k)-\frac{\rho^2(k+\ell)}{4\overline{\rho(k)}}\right.$

$$\left.\left(h(t,k)-\frac{\Delta_\ell\rho(k)}{\rho(k+\ell)}\sqrt{H(t,k)}\right)^2\right]$$

$$< H(t,k_2)w(k_2)\le H(t,0)w(k_2),$$

which implies that

$$\sum_{k=0}^{t-1}\left[H(t,k)\rho(k)q(k)\right.$$

$$\left.-\frac{\rho^2(k+\ell)}{4\overline{\rho(k)}}\left(h(t,k)-\frac{\Delta_\ell\rho(k)}{\rho(k+\ell)}\sqrt{H(t,k)}\right)^2\right]$$

$$< H(t,0)\sum_{k=0}^{k_2-1}\rho(k)q(k+\ell)+H(t,0)w(k_2).$$

Hence

$$\limsup_{t\to\infty}\frac{1}{H(t,0)}\sum_{k=0}^{t-1}\left[H(t,k)\rho(k)q(k)-\right.$$

$$\left.\frac{\rho^2(k+\ell)}{4\overline{\rho(k)}}\left(h(t,k)-\frac{\Delta_\ell\rho(k)}{\rho(k+\ell)}\sqrt{H(t,k)}\right)^2\right]<\infty$$

which is contrary to (24). The proof is complete.

By choosing the function $H(t,k)$ in appropriate manners, we can derive several oscillation criteria for (1). For instance, let us consider the two valuable function $H(t,k)$ defined by $H(t,k)=(t-k)^\lambda$, or $H(t,k)=\left(\log\left(\frac{t+\ell}{k+\ell}\right)\right)^\lambda$, where $\lambda\ge 1$, $t\ge k\ge 0$. Then $H(t,t)=0$ for $t\ge 0$ and $H(t,k)>0$ and $\Delta_{\ell(k)}H(t,k)\le 0$ for $t>k>0$. Hence we have the following results.

Corollary 3.6 Assume that all the assumptions of Theorem 3.5 hold, except the condition (24) is replaced by

$$\limsup_{t\to\infty}\frac{1}{t^\lambda}\sum_{k=0}^{t}\left[(t-k)^\lambda\rho(k)Q(k)-\frac{\rho^2(k+\ell)}{4\overline{\rho(k)}}\right.$$

$$\left.\left(\lambda(t-k)^{\frac{\lambda}{2}-1}-\frac{\Delta_\ell\rho(k)}{\rho(k+\ell)}\sqrt{(t-k)^\lambda}\right)^2\right]=\infty,$$

where $Q(k)=q(k)(1-p(k-\sigma\ell))^\gamma$. Then every solution of equation (1) oscillates.

Corollary 3.7 Assume that all the assumptions of Theorem 3.5 hold, except the condition (24) is replaced by

$$\limsup_{t\to\infty}\frac{1}{(\log(t+\ell))^\lambda}\sum_{k=0}^{t}\left[\left(\log\left(\frac{t+\ell}{k+\ell}\right)\right)^\lambda\right.$$

where

$$\left.\rho(k)Q(k)-\frac{\rho^2(k+\ell)}{4\overline{\rho(k)}}B(t,k)\right]=\infty,$$

$$B(t,k)=\left(\frac{\lambda}{k+\ell}\left(\ln\frac{t+\ell}{k+\ell}\right)^{\frac{\lambda}{2}-1}\right.$$

$$\left.-\frac{\Delta_\ell\rho(k)}{\rho(k+\ell)}\sqrt{\left(\ln\frac{t+\ell}{k+\ell}\right)^\lambda}\right)^2.$$

Then every solution of equation (1) oscillates.

Another H(t,k) may be chosen as

$$H(t,k)=\phi(t,k),\qquad t\ge k\ge 0,$$

$$H(t,k)=(t-k)_\ell^{(\lambda)},\quad \lambda>2,\ t\ge k\ge 0,$$

where $\phi:[0,\infty)\to[0,\infty)$ is a continuously differentiable function which satisfies $\phi(0)=0$ and $\phi(x)>0,\phi'(x)$ for $x>0,$ and $(t-k)_\ell^{(\lambda)}=(t-k)(t-k-\ell)\dots(t-k-(\lambda-1)\ell)$ and

$$\Delta_{\ell(k)}(t-k)_\ell^{(\lambda)}=(t-k-\ell)^{(\lambda)}-(t-k)_\ell^{(\lambda)}$$

$$=-\lambda\ell(t-k)_\ell^{(\lambda-1)}.$$

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V.Chandrasekar /Department of Mathematics/
S.K.P. Engineering College,Tiruvannamalai/Tamil Nadu/S.India.
drchanmaths@gmail.com.
A.Benevatho Jaison /Department of Mathematics/
S.K.P. Engineering CollegeTiruvannamalai/ Tamil Nadu/ S.India.
benejaison@gmail.com