OSCILLATION THEOREMS FOR SECOND ORDER GENERALIZED LINEAR DIFFERENCE EQUATIONS WITH DAMPING TERM

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Abstract: In this paper, the authors present new oscillation criteria of the generalized second order difference equation with damping

$\Delta_\ell(\gamma(\mathrm{k})\Delta_\ell u(\mathrm{k}))+\mathrm{p}(\mathrm{k})\,\mathrm{u}(\mathrm{k})=0, \quad \mathrm{k}\in[\mathrm{k}_0,\infty)$

where $p(k)$ and $q(k)$ are allowed to change sign on $[k_0,\infty)$, and $\gamma(k)>0$. One of our results in new even for the difference equations $\Delta_\ell{}^2 u(\mathrm{k})+\mathrm{q}(\mathrm{k})\,\mathrm{u}(\mathrm{k})=0,$

and $\Delta_\ell{}^2 u(\mathrm{k})+p(\mathrm{k})\Delta_\ell u(\mathrm{k})+\mathrm{q}(\mathrm{k})\,\mathrm{u}(\mathrm{k})=0.$

Keywords: Damping Term, Generalized Difference Equation, Oscillation.

Introduction: Difference equation usually describes the evaluation of some certain phenomena over time and is also important in describing dynamics for fundamentally discrete system, see [1]. For example, in the numerical integration, the standard approach is to use the difference equation, similarly, the population dynamics have discrete generations; the size of the $(k+1)^{th}$ generation $u(k+1)$ is a function of the k^{th} generation $u(\mathrm{k})$ This can be expressed as difference equation of the form

$f(u(k+1))=f(u(k))$

See the example [8]. Further, the concepts of difference equations with many examples in application such as asymptotic behavior of solution of difference equations were studied extensively by Elayadi [8] where the analytic and geometric approaches were also combined in order to studying difference equations. Further, in [8], both classical modern treatments of the difference equations were presented in excellent form. For related results on difference equations, see [3] - [6].

The basic theory of difference equations is based on the difference operator Δ is defined as

$\Delta u(k)=u(k+1)-u(k), k\in \mathrm{N}$ (1)

where $\square=\{0,1,2,...\}$. Even though many authors have suggested the definition of Δ is defined as

$\Delta u(k)=u(k+\ell)-u(k), k\in \mathrm{N},$

$\ell\in[0,\infty)$ (2)

no significant progress took place on this line. But recently, when we took up the definition of Δ as given in the (3) and developed the theory of difference equation in a different direction and obtained some interesting result in the application of number theory. For convenience, the difference operator Δ is defined as (2) is labelled as Δ_ℓ and by defining Δ_ℓ and its inverse Δ_ℓ^{-1} many interesting results and applications in number theory as well as in fluid dynamics can be obtained. By extending the study for sequences of complex numbers and ℓ to be real, some new qualitative properties like rotatory, expending, shrinking, spiral and weblike structures were studied for the solutions of difference equations involving for similar results, refer to [3] - [6]. Hence, in this paper, we study the oscillatory behavior of the solution of the second order difference equation with damping

$\Delta_\ell(\gamma(k)\Delta_\ell u(k))+p(k)\Delta_\ell u(k) \quad +q(k)u(k)=0,$

(3)

where γ, p and q are continuous on

$[k_0,\infty), k_0>0, \gamma>0,$ and p and q are allowed to take on negative values for arbitrarily large k. The oscillatory character is considered in the usual sense, i.e., A solution $u(k)$ of equation (3) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called non-oscillatory. Equation (3) is called oscillatory if all its solutions are oscillatory.

In the absence of damping, there is a very large body of literature devoted to the corresponding equations

$\Delta_\ell^2 u(k)+q(k)u(k)=0,$ (4)

$\Delta_\ell(\gamma(\mathrm{k})\Delta_\ell u(\mathrm{k}))+\mathrm{q}(\mathrm{k})\,\mathrm{u}(\mathrm{k})=0.$ (5)

Although (3) can be put in the forms (4) and (5) by multiplication by an integrating factor and, if necessary, by simple transformations, there are advantages in obtaining direct oscillation theorems for (3): besides the obvious practical advantage of eliminating the need for the integrating factor, there is an incentive in developing methods which will generalize to more general equations. Throughout this paper, we make use of the following assumptions:

(i) $\mathrm{N}(1)=\{1,2,3,\cdots\}$.

(ii) k means by $k\ell+j$.

2. Main Result and Examples

Definition 2.1 [2] Let $n\in \mathrm{N}(1)$. Then the generalized polynomial factorial is defined by

$k_\ell^{(n)}=k(k-\ell)(k-2\ell)\cdots(k-(n-1)\ell).$

Lemma 2.2 [3] If n is positive integer, then

$\Delta_\ell k_\ell^{(n)} = n\ell k_\ell^{(n-1)}$.

Theorem 2.3 If there exist $\eta \in N(1)$, $\beta \in [0,\ell)$ such that

$$\lim_{k\to\infty}\sup k^{-\eta}\sum_{\tau=k_0}^{k}(k-\tau\ell-j)_\ell^{(\eta)}(\tau\ell+j)^{\beta} q(\tau\ell+j)=\infty, \quad (6)$$

$$\lim_{k\to\infty}\sup k^{-\eta}\sum_{\tau=k_0}^{k}\Big\{\big[(k-(\tau\ell+j))_\ell^{(n)}p(\tau\ell+j)(\tau\ell+j)+\eta\ell(\tau\ell+j)-\beta(k-(\tau\ell+j))_\ell^{(n)}\big]^2 (k-(\tau\ell+j))_\ell^{(\eta-2)}(\tau\ell+j)^{\beta-2}\Big\}<\infty, \quad (7)$$

then (3) is oscillatory.

Of particular interest, therefore, is the problem of finding criteria for the oscillation of (3) when (6) or (7) is not satisfied. In this paper we will establish two oscillation theorems. The first theorem considerably improves some known results and the second is new even for equations (4) and (5). Our results are as follows:

Theorem 2.4 Suppose that there exist a positive real valued function $h(k)$ on $[k_0,\infty)$, $\omega(k)$ is nonincreasing function and a constant $\eta \in N(1)$ such that

$$\lim_{k\to\infty}\sup k^{-\eta}\sum_{\tau=k_0}^{k}\Big\{(k-(\tau\ell+j))_\ell^{(\eta)}h(\tau\ell+j)q(\tau\ell+j) -\frac{1}{4}\Big[(k-(\tau\ell+j)-(\eta-1)\ell)_\ell^{(2)}\frac{h(\tau\ell+j)p(\tau\ell+j)}{\gamma(\tau\ell+j)} +\eta\ell h(\tau\ell+j+\ell)-(k-(\tau\ell+j)-(\eta-1)\ell)_\ell^{(2)} \quad (8)$$
$$\Delta_\ell h(\tau\ell+j)\Big]^2(k-(\tau\ell+j))_\ell^{(\eta-2)}\frac{\gamma(\tau\ell+j)}{h(\tau\ell+j)}\Big\}=\infty$$

then equation (3) is oscillatory.

Proof. Let $u(k)$ be a nonoscillatory solution of (3). Without loss of generality, we suppose that $u(k)\neq 0$ for $k\ge k_0$. Furthermore, we put

$$\omega(k)=\gamma(k)\Delta_\ell u(k)/u(k).$$

Then it follows from (3) that

$$\Delta_\ell\omega(k)+\frac{\omega(k)\omega(k+\ell)}{\gamma(k)}+\frac{p(k)\omega(k)}{\gamma(k)}+q(k)=0,\quad k\ge k_0, \quad (9)$$

and consequently, for all $k > s \ge k_0$,

$$\sum_{\tau=s}^{k}(k-(\tau\ell+j))_\ell^{(\eta)}h(\tau\ell+j)\Delta_\ell\omega(\tau\ell+j) +\sum_{\tau=s}^{k}\frac{h(\tau\ell+j)\omega(\tau\ell+j)\omega((\tau+1)\ell+j)}{\gamma(\tau\ell+j)}(k-(\tau\ell+j))_\ell^{(\eta)} +\sum_{\tau=s}^{k}(k-(\tau\ell+j))_\ell^{(\eta)}\frac{h(\tau\ell+j)p(\tau\ell+j)\omega(\tau\ell+j)}{(\tau\ell+j)}+\sum_{\tau=s}^{k}(k-(\tau\ell+j))_\ell^{(\eta)}h(\tau\ell+j)q(\tau\ell+j)=0.$$

Since

$$\sum_{\tau=s}^{k}(k-(\tau\ell+j))_\ell^{(\eta)}h(\tau\ell+j)\Delta_\ell\omega(\tau\ell+j) =-(k-(s\ell+j))_\ell^{(\eta)}h(s\ell+j)\omega(s\ell+j)-\sum_{\tau=s}^{k}\omega((\tau+1)\ell+j)(k-(\tau\ell+j))_\ell^{(\eta)}\Delta_\ell h(\tau\ell+j) -\eta\ell\sum_{\tau=s}^{k}(k-(\tau\ell+j))_\ell^{(\eta-1)} h((\tau+1)\ell+j)\omega((\tau+1)\ell+j),$$

we obtain that

$$\sum_{\tau=s}^{k}(k-(\tau\ell+j))_\ell^{(\eta)}h(\tau\ell+j)q(\tau\ell+j) =-\sum_{\tau=s}^{k}\frac{(k-(\tau\ell+j))_\ell^{(\eta)}h(\tau\ell+j)\omega(\tau\ell+j)}{\gamma(\tau\ell+j)}\omega((\tau+1)\ell+j)+(k-(s\ell+j))_\ell^{(\eta)}h(s\ell+j)\omega(s\ell+j)-\sum_{\tau=s}^{k}\Big[(k-(\tau\ell+j)-(\eta-1)\ell)\frac{h(\tau\ell+j)p(\tau\ell+j)}{\gamma(\tau\ell+j)}+\eta\ell h((\tau+1)\ell+j) -(k-(\tau\ell+j)-(\eta-1)\ell)\Delta_\ell h(\tau\ell+j)\Big](k-(\tau\ell+j))_\ell^{(\eta-1)}\omega(\tau\ell+j),$$

$$\sum_{\tau=s}^{k}(k-(\tau\ell+j))_{\ell}^{(\eta)}h(\tau\ell+j)q(\tau\ell+j)$$
$$\leq(k-(s\ell+j))_{\ell}^{(\eta)}h(s\ell+j)\omega(s\ell+j)-$$
$$\sum_{\tau=s}^{k}(k-(\tau\ell+j)-(\eta-1)\ell)\frac{\omega^2(\tau\ell+j)}{\gamma(\tau\ell+j)}$$
$$h(\tau\ell+j)-\sum_{\tau=s}^{k}[(k-(\tau\ell+j)-(\eta-1)\ell)$$
$$h(\tau\ell+j)\frac{p(\tau\ell+j)}{\gamma(\tau\ell+j)}+\eta\ell h(\tau\ell+j)$$
$$-(k-(\tau\ell+j)-(\eta-1)\ell)\Delta_\ell h(\tau\ell+j)]$$
$$(k-(\tau\ell+j))_{\ell}^{(\eta-1)}\omega(\tau\ell+j) \quad (10)$$

and hence

$$\sum_{\tau=s}^{k}(k-(\tau\ell+j))_{\ell}^{(\eta)}h(\tau\ell+j)q(\tau\ell+j)$$
$$\leq(k-(s\ell+j))_{\ell}^{(\eta)}h(s\ell+j)\omega(s\ell+j)$$
$$-\sum_{\tau=s}^{k}(k-(\tau\ell+j)-(\eta-1)\ell)_{\ell}^{(\eta)}$$
$$\frac{h(\tau\ell+j)\omega^2(\tau\ell+j)}{\gamma(\tau\ell+j)}-$$
$$\sum_{\tau=s}^{k}\left[\eta\ell h(k+\tau\ell+j)+(k-(\eta-1)\ell)\Delta_\ell h(\tau\ell+j)\right]$$
$$\frac{(k-(\eta-1)\ell)h(\tau\ell+j)\omega(\tau\ell+j)}{\gamma(\tau\ell+j)}-$$
$$\sum_{\tau=s}^{k}(k-(\tau\ell+j))_{\ell}^{(\eta)}\frac{h(\tau\ell+j)p(\tau\ell+j)\omega(\tau\ell+j)}{\gamma(\tau\ell+j)}$$
$$\leq(k-(s\ell+j))_{\ell}^{(\eta)}h(s\ell+j)\omega(s\ell+j),\ s\geq k_0. \quad (11)$$

Divide (11) by $k_{\ell}^{(\eta)}$ and take the upper limit as $k\to\infty$. Using (8), we obtain a contradiction. This completes the proof of the theorem.

Corollary 2.5 *If*

$$\limsup_{k\to\infty}k^{-2}\sum_{\tau=k_0}^{k}\Big[(k-(\tau\ell+j))_{\ell}^{(2)}h(\tau\ell+j)q(\tau\ell+j)$$
$$-\frac{(k-(\tau\ell+j))_{\ell}^{(2)}}{4h(\tau\ell+j)}\Delta_\ell h^2(\tau\ell+j)\gamma(\tau\ell+j)-$$
$$(k-(\tau\ell+j))h(\tau\ell+j)\Delta_\ell\gamma(\tau\ell+j)\Big]=\infty \quad (12)$$

with $h(k)$ as in Theorem 2.4, then equation (7) is oscillatory.

Proof. From (9), letting $p(k)\equiv 0$ and $\eta=2$, we have that

$$k^{-2}\sum_{\tau=k_0}^{k}\Big[(k-(\tau\ell+j))_{\ell}^{(2)}h(\tau\ell+j)q(\tau\ell+j)-$$
$$h(\tau\ell+j)\gamma(\tau\ell+j)-\frac{(k-(\tau\ell+j))_{\ell}^{(2)}\gamma(\tau\ell+j)}{4h(\tau\ell+j)}$$
$$\Delta_\ell h^2(\tau\ell+j)-(k-(\tau\ell+j))\Delta_\ell h(\tau\ell+j)$$
$$\gamma(\tau\ell+j)\leq k^{-2}(k-k_0)^2h(k_0)\omega(k_0). \quad (13)$$
$$\sum_{\tau=k_0}^{k}(k-(\tau\ell+j))\Delta_\ell h(\tau\ell+j)\gamma(\tau\ell+j)$$

Since,

$$=\sum_{\tau=k_0}^{k}\sum_{\tau=k_0}^{\tau}\Delta_\ell h(\xi)\gamma(\xi)$$
$$=\sum_{\tau=k_0}^{k}h(\tau\ell+j)\gamma(\tau\ell+j)-\sum_{\tau=k_0}^{k}(k-(\tau\ell+j))$$
$$h(\tau\ell+j)\Delta_\ell\gamma(\tau\ell+j)-h(t_0)\gamma(k_0)(k-k_0) \quad (14)$$

(15) and (16) together contradict our hypothesis (14).

Theorem 2.6 Suppose that there exist a positive real valued function $h(k)$ on $[k_0,\infty)$ and $\eta\in N(1)$ such that

$$\limsup_{k\to\infty}k^{-\eta}\sum_{\tau=k_0}^{k}(k-(\tau\ell+j))_{\ell}^{(\eta)} \quad (15)$$
$$h(\tau\ell+j)q(\tau\ell+j)<\infty,$$

and there exists a continuous function $\varphi(k)$ on $[k_0,\infty)$ such that

$$\liminf_{k\to\infty}k^{-\eta}\sum_{\tau=s}^{k}\Big\{(k-(\tau\ell+j))_{\ell}^{(\eta)}h(\tau\ell+j)$$
$$q(\tau\ell+j)-\frac{1}{4}\Big[(k-(\tau\ell+j)-(\eta-1)\ell)_{\ell}^{(2)}$$
$$\frac{h(\tau\ell+j)}{\gamma(\tau\ell+j)}p(\tau\ell+j)+\eta\ell h(\tau+\ell j)$$
$$-(k-(\tau\ell+j)-(\eta-1)\ell)_{\ell}^{(2)}\Delta_\ell h((\tau+1)\ell+j)\Big]^2\Big\}$$
$$\left.(k-(\tau\ell+j))_{\ell}^{(\eta-2)}\frac{\gamma(\tau\ell+j)}{h(\tau\ell+j)}\right\}\geq\varphi(s), \quad (16)$$

and

$$\lim_{k\to\infty}\sum_{\tau=k_0}^{k}\frac{\varphi_+^2(\tau\ell+j)}{h(\tau\ell+j)\gamma(\tau\ell+j)}=\infty, \quad (17)$$

where $\varphi_+(k)=\max(\varphi(k),0)$, then equation (3) is oscillatory.

***Proof*.** Suppose that $u(k)$ is a solution of (3) with $u(k)\neq 0$ for $k\geq k_0$. Set $\omega(k)=\gamma(k)\Delta_\ell u(k)/u(k)$. As in the proof of Theorem 2.4, (9) holds. Dividing (9) by

$k_\ell^{(\eta)}$ and taking the lower limit as $k \to \infty$, we obtain $\varphi(s) \le h(s)\omega(s),\ s \ge k_0$, which implies that

$$\varphi_+^2(s) \le h^2(s)\omega^2(s). \qquad (18)$$

For $k > k_0$, we define the functions

$$x(k) = k^{-\eta}\sum_{\tau=k_0}^{k}\Big[(k-(\eta-1)\ell)\frac{h(\tau\ell+j)p(\tau\ell+j)}{\gamma(\tau\ell+j)} +\eta\ell(\tau\ell+j) - (k-(\eta-1)\ell)\Delta_\ell h(\tau\ell+j)\Big]^2 (k-(\tau\ell+j))_\ell^{(\eta-1)}\omega(\tau\ell+j),$$

$$y(k) = k^{-\eta}\sum_{\tau=k_0}^{k}(k-(\tau\ell+j))_\ell^{(\eta)}h(\tau\ell+j)\frac{\omega(\tau\ell+j)\omega((\tau+1)\ell+j)}{\gamma(\tau\ell+j)}.$$

From (8),

$$x(k)+y(k) = k^{-\eta}(k-k_0)_\ell^{(\eta)}h(k_0)\omega(k_0) - k^{-\eta}\sum_{\tau=k_0}^{k}(k-(\tau\ell+j))_\ell^{(\eta)}h(\tau\ell+j)q(\tau\ell+j) \qquad (19)$$

and we observe that (16) implies for $s \ge k_0$,

$$\liminf_{k\to\infty} k^{-\eta}\sum_{\tau=k_0}^{k}(k-(\tau\ell+j))_\ell^{(\eta)} h(\tau\ell+j)q(\tau\ell+j) \ge \varphi(s), \qquad (20)$$

and

$$\limsup_{k\to\infty} k^{-\eta}\sum_{\tau=k_0}^{k}(k-(\tau\ell+j))_\ell^{(\eta)}h(\tau\ell+j) q(\tau\ell+j) - \liminf_{k\to\infty}\frac{k^{-\eta}}{4} \sum_{\tau=k_0}^{k}\Big[(k-(\tau\ell+j)-(\eta-1)\ell)_\ell^{(2)}\frac{h(\tau\ell+j)}{\gamma(\tau\ell+j)} p(\tau\ell+j)+\eta\ell h((\tau+1)\ell+j) -(k-(\tau\ell+j)-(\eta-1)\ell)_\ell^{(2)}h(\tau\ell+j)\Big]^2$$
$$\times(k-(\tau\ell+j))_\ell^{(\eta-2)}\frac{\gamma(\tau\ell+j)}{h(\tau\ell+j)} \ge \varphi(k_0). \qquad (21)$$

(21) together with (15) shows that there exists a sequence $\{k_n\}_1^\infty, k_n > k_0,\ n = 1,2,3,\cdots, \lim_{n\to\infty} k_n = \infty,$ (22)

such that

$$\lim_{k\to\infty}\frac{k_n^{-\eta}}{4}\sum_{\tau=k_0}^{k_n}\Big[(k_n-(\tau\ell+j)-(\eta-1)\ell)_\ell^{(2)} \frac{h(\tau\ell+j)p(\tau\ell+j)}{\gamma(\tau\ell+j)}+\eta\ell((\tau+1)\ell+j) - (k-(\tau\ell+j)-(\eta-1)\ell)_\ell^{(2)}\Delta_\ell h(\tau\ell+j)\Big]^2 (k_n-(\tau\ell+j))_\ell^{(\eta-2)}\frac{\gamma(\tau\ell+j)}{h(\tau\ell+j)} < \infty. \qquad (23)$$

Taking the upper limit as $k \to \infty$ in (19) and using (20), we have

$$\limsup_{k\to\infty} x(k)+y(k) = -\liminf_{k\to\infty} k^{-\eta}\sum_{\tau=k_0}^{k}h(\tau\ell+j) (k-(\tau\ell+j))_\ell^{(\eta)}q(\tau\ell+j)h(k_0)\omega(k_0) = h(k_0)\omega(k_0)-\varphi(k_0) = k. \qquad (24)$$

Hence for all sufficiently large n,

$$x(k_n)+y(k_n) < k. \qquad (25)$$

Since

$$y(k) = \sum_{\tau=k_0}^{k}k^{-\eta}(k-(\tau\ell+j))_\ell^{(\eta)}h(\tau\ell+j) \frac{\omega(\tau\ell+j)\omega((\tau+1)\ell+j)}{\gamma(\tau\ell+j)} > 0$$

is increasing in $k > k_0$, we see that $\lim_{k\to\infty} y(k) = c$, where $c = \infty$ or is a positive constant. Suppose that $c = \infty$, then $\lim_{k\to\infty} y(k_n)$ and, by (25),

$$\lim_{k\to\infty} x(k_n) = -\infty. \qquad (26)$$

(25) and (26) lead to $x(k_n)/y(k_n)+1 < \varepsilon$, where $0 < \varepsilon < 1$ is a constant, that is,

$x(k_n)/y(k_n) < \varepsilon - 1 < 0,$ for all large k_n. (27)

One the other hand, by the Schwarz inequality we have

$$0 \le k_n^{-2\eta}\Bigg(\sum_{\tau=k_0}^{k_n}\Big[(k_n-(\tau\ell+j)-(\eta-1)\ell) \frac{h(\tau\ell+j)p(\tau\ell+j)}{\gamma(\tau\ell+j)}+\eta\ell((\tau+1)\ell+j) -(k_n-(\tau\ell+j)-(\eta-1)\ell)\Delta_\ell h(\tau\ell+j)\Big] (k_n-(\tau\ell+j))_\ell^{(\eta-1)}\omega(\tau\ell+j)\Bigg)^2$$

$$\le k_n^{-\eta}\sum_{\tau=k_0}^{k_n}\Big[(k_n-(\tau\ell+j)-(\eta-1)\ell)_\ell^{(2)}$$

$$\frac{h(\tau\ell+j)p(\tau\ell+j)}{\gamma(\tau\ell+j)}+\eta\ell(\tau\ell+j)$$
$$-(k_n-(\tau\ell+j)-(\eta-1)\ell)_\ell^{(2)}\Delta_\ell h(\tau\ell+j)\Big]^2$$
$$(k_n-(\tau\ell+j))_\ell^{(\eta-2)}\frac{\gamma(\tau\ell+j)}{h(\tau\ell+j)}$$
$$\left(k_n^{-\eta}\sum_{\tau=k_0}^{k_n}(k_n-(\tau\ell+j)-(\eta-1)\ell)_\ell^{(2)}\right.$$
$$\left.h(\tau\ell+j)\frac{\omega(\tau\ell+j)\omega((\tau+1)\ell+j)}{\gamma(\tau\ell+j)}\right),$$

for all large k_n, and so

$$0\le\frac{x^2(k_n)}{y(k_n)}$$
$$\le k_n^{-\eta}\sum_{\tau=k_0}^{k_n}\Big[(k_n-(\tau\ell+j)-h(\tau\ell+j)(\eta-1)\ell)_\ell^{(2)}$$
$$\frac{p(\tau\ell+j)}{\gamma(\tau\ell+j)}+\eta\ell((\tau+1)\ell+j)$$
$$-(k_n-(\tau\ell+j)-(\eta-1)\ell)_\ell^{(2)}\Delta_\ell h(\tau\ell+j)\Big]^2$$
$$(k_n-(\tau\ell+j))_\ell^{(\eta-2)}\frac{\gamma(\tau\ell+j)}{h(\tau\ell+j)}.$$

By (25), we have

$$0\le\lim_{k\to\infty}\frac{x^2(k_n)}{y(k_n)}$$
$$\le\lim_{k\to\infty}k_n^{-\eta}\sum_{\tau=k_0}^{k_n}\Big[(k_n-(\tau\ell+j)-(\eta-1)\ell)_\ell^{(2)}$$
$$\frac{h(\tau\ell+j)p(\tau\ell+j)}{\gamma(\tau\ell+j)}+\eta\ell((\tau+1)\ell+j)-$$
$$(k_n-(\tau\ell+j)-(\eta-1)\ell)_\ell^{(2)}\Delta_\ell h(\tau\ell+j)]^2$$
$$(k_n-(\tau\ell+j))_\ell^{(\eta-2)}\frac{\gamma(\tau\ell+j)}{h(\tau\ell+j)}<\infty.$$

which contradicts (28) and (27).

Hence $\lim_{k\to\infty}y(k)=c<\infty$. Using (20), we then obtain that

$$\lim_{k\to\infty}k^{-\eta}\sum_{\tau=k_0}^{k}(k-(\tau\ell+j))_\ell^{(\eta)}\frac{\varphi_+^2(\tau\ell+j)}{h(\tau\ell+j)\gamma(\tau\ell+j)}$$
$$\le\lim_{k\to\infty}k^{-\eta}\sum_{\tau=k_0}^{k}(k-(\tau\ell+j))_\ell^{(\eta)}h(\tau\ell+j)$$
$$\frac{\omega(\tau\ell+j)\omega((\tau+1)\ell+j)}{\gamma(\tau\ell+j)}=c$$

which contradicts condition (19). This completes the proof of Theorem 2.6.

Remark 2.7 In the conditions of Theorem 2.6, $q(k)$ is not required to be sumable or bounded on $[k_0,\infty)$. See Examples 2.8 and 2.9 below.

Example 2.8 Consider the equation

$$\Delta_\ell\left(\frac{1}{k}\Delta_\ell u(k)\right)+\sin k\Delta_\ell u(k)$$
$$+k^2\cos ku(k)=0,\quad k\ge k_0>0,\ (28)$$

If we take $h(k)=k$ and $\eta=2$, then all the hypotheses of Theorem 2.4 are satisfied. Hence equation (28) is oscillatory.

Example 2.9 *Consider the equation*

$$\Delta_\ell(k_\ell^{(\lambda)}\Delta_\ell u(k))+k^\mu\sin k\Delta_\ell u(k)$$
$$+k^v\cos k\,u(k)=0,\ k\ge k_0>0,\quad (29)$$

where $\lambda\in\mathrm{N}(1),\infty<\mu\le1$ and $1<v\le1$ are constants and $2v+1\ge\lambda$.

Taking $h(k)=1$ and $\eta=2$, we have

$$\limsup_{k\to\infty}k^{-2}\sum_{\tau=s}^{k}(k-(\tau\ell+j))_\ell^{(2)}(\tau\ell+j)^v$$
$$\cos(\tau\ell+j)=-k_0^v\sin k_0<\infty,$$
$$\liminf_{k\to\infty}k^{-2}\sum_{s}^{k}(k-(\tau\ell+j))_\ell^{(2)}(\tau\ell+j)^v\cos(\tau\ell+j)$$
$$-\frac{[(k-(\tau\ell+j))(\tau\ell+j)^\mu\sin(\tau\ell+j)+2]^2}{4}$$
$$(\tau\ell+j)^\lambda\ \ge-(\mathrm{s}\,\ell+j)^v\sin(\mathrm{s}\,\ell+j)-t,\ \ s\ge k_0,$$

where t is a positive constant. Set $\varphi(s)=-s^\upsilon\sin st$, there is an integer N such that $(2N+1)\pi+\pi/4>k_0$, and when $n\ge N$ and $(2n+1)\pi+\pi/4\le s\le2(n+1)\pi-\pi/4$, $\varphi(s)=-s^v\sin s-t\ge\varepsilon s^v$, where ε is a small constant. Noting $2v-\lambda\ge-1$, we have

$$\lim_{k\to\infty}\sum_{k_0}^{k}\frac{\varphi_+^2(s)}{s^\lambda}\ge\sum_{N=n}^{\infty}\varepsilon^2\sum_{(2n+1)\pi+\pi/4}^{(2n+1)\pi-\pi/4}s^{2v-\lambda}$$
$$\ge\sum_{N=n}^{\infty}\varepsilon^2\sum_{(2n+1)\pi+\pi/4}^{(2n+1)\pi-\pi/4}\frac{1}{s}=\infty.$$

Hence (29) is oscillatory by Theorem 2.6, whereas known of the none criteria can cover this result.

References

1. R.P.Agarwal, Differences Equations and Inequalities, Marcel Dekkar, New Yark, 2000.
2. V. Kamenev, "Integral Criterion for Oscillations of Linear Differential Equations of Second Order", Mat. Zametki 33 (1978), 249-251.
3. M.Maria Susai Manuel, G.Britto Antony Xavier and E.Thandapani, "Theory of Generalized Difference Operator and its Applications", For East Journal of Mathematical sciences, 20(2), 2006, 163-17.
4. M.Maria Susai Manuel, G.Britto Antony Xavier and E.Thadapani, "Qualitative Properties of Solutions of Certain Class of Difference Equations", For East Journal of Mathematical Sciences, 23(3) (2006), 295-304.
5. M.Maria Susai Manuel, G.Britto Antony Xavier and E.Thadapani, "Generalized Bernoulli Polynomials Through Weighted Pochhammer Symbols", Far East Journal of Applied Mathematics, 26(3), (2007), 321-333.
6. M.Maria Susai Manuel, G.Britto Antony Xavier, "Recessive Dominant and Spiral Behaviours of Solutions o Certain Class of Difference Equations", International Journal of Difference Equations and Applications, 10(4) (2007), 423-433.
7. Ronald E.Mickens, Difference Equations, Van Nostrand Reinhold Company, New Yark, 1990.
8. Saber N. Elayadi, An Introduction to Difference Equations, Third Edition, Spinger, USA, 2000.
9. Xiaoping Li, Jianchu Jiang, "Oscillation of Second-Order Linear Difference Equations", Mathematical and Computer Modeling, 35 (2002), 983-990.
10. Jurang Yan, "A note on an oscillation criterion for an equation with damped term", Proc. Amer. Math. Soc., 90 (1984), 277-280.
11. B.G.Zhang and S.S.Cheng, "Oscillation Criteria and Comparison Theorems for Delay Difference Equations", Fasciculi Mathematici, 25 (1995), 13-32.
12. Y.Zhou and B.G.Zhang, "Oscillation of Delay Difference Equations in a Critical State", Computers Math. Applic. 39 (7/8), 71-80, 2000.

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