ON NUMBER OF RESTRICTED PARTITIONS WITH A CONGRUENCE CONDITION

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Abstract: Let $R_k(n,r,m)$ be the number of partitions of a positive integer n with exactly k parts and least part congruent to r modulo m, where $m > 1$ and $0 \le r \le m-1$. This article is concerned with finding an asymptotic estimate of $R_k(n,r,m)$ and Ramanujan type congruences for $R_k(n,r,m)$. All the findings in this article is based upon the fact that, $R_k(n,r,m)$is a quasi polynomial.

Keywords: Estimates, Partitions, Quasi polynomials, Ramanujan type congruences.

Introduction: By partition of a positive integer n, we mean a sequence of non increasing positive integers say $\lambda = (a_1, \cdots, a_m)$ such that $\sum_{i=1}^{m} a_i = n$. Each a_i is called a part of the partition λ. Let $p(n)$be the number of partitions of n. Following asymptotic estimate of $p(n)$ is well known: $p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$

This estimate is an outcome of the convergent series expansion given by Ramanujan [7, 13] for $p(n)$.

In sequel of this, many authors have defined several kinds of partitions by imputing congruence conditions over the parts and ensued to find convergent series representation and asymptotic estimate of its enumerative functions. Using circle dissection method of Hardy, Lehner and Livingood [9,10] obtained an asymptotic formula for $p_a(n)$, where $p_a(n)$ is defined to be the number of partitions of n into parts congruent to $\pm a_i (mod\ p)$; here a_i belongs to a subset of $\left\{1, 2, \cdots, \frac{p-1}{2}\right\}$ where p is an odd prime. The other works done in this direction can be seen in papers by Emil Grosswald [6], Iseki [8], Peter Hagis [11,12], M. M. Robertson and D. Spencer [14], and Subramanyasastri [15].

Motivated by this, we define a new kind of partition namely $least\ r - modulo\ - m\ partition$.

Definition 1.1. Let n be a positive integer and let $m > 1$ and r be integers with $0 \le r \le m-1$. A partition say $\lambda = (a_1, a_2, \cdots, a_k)$ of n is said to be $least\ r - modulo\ - m\ partition$ of n if its least part $a_k \equiv r(mod\ m)$. The number of least r-modulo-m partitions of n with exactly k parts is denoted by $R_k(n,r,m)$.

Definition 1.2. An arithmetical function f is said to be a $Quasi\ polynomial$ if, $f(\alpha l + r)$ is a polynomial in l for each $r = 0, 1, \cdots, \alpha - 1$, where α is a positive integer greater than 1. Each polynomial $f(\alpha l + r)$ is called constituent polynomial of f and α is called quasi period of f.

The rest of this article is organized as follows: Section 2 is concerned with deriving a recurrence relation of $R_k(n,r,m)$; this recurrence relation is instrumental for further investigation of the subject. In section 3, we find an asymptotic estimate of $R_k(n,r,m)$ whose derivation is based on the fact that $R_k(n,r,m)$ is a quasi polynomial with quasi period $(mn)!$ and each constituent polynomial being of degree $k-1$ with identical leading coefficients. In section 4, a method is depicted for the derivation of Ramanujan type congruences for a function $f(n)$ provided it is a quasi polynomial; this method is applied to give Ramanujan type congruences for the function $R_k(n,r,m)$ for the initial case: $k = 3, m = 2$ and $r = 0$.

2. A recurrence relation of $R_k(n,r,m)$

In this section, we find a recurrence relation satisfied by $R_k(n,r,m)$. This recurrence relation is crucial for the forth coming results of this paper.

Lemma 2.1. We have

$R_k(n,r,m) = R_k(n-km,r,m) + \sum_{i=1}^{k-1} p(n-rk,i)$

where $p(n,k)$ is defined to be the number of partitions of n with exactly k parts with $p(0,k) = 1$ and $p(n,k) = 0$ when $n < 0$.

Proof. Let $\lambda = (a_1, \cdots, a_k)$ be a least r-modulo-m partition of n.

Case(i). Assume that $a_k > r$. Since $a_k \equiv r(mod\ m)$, we have $a_k = qm + r$ for some positive integer q. As a consequence of this, we get $a_k - m \equiv r(mod\ m)$.

Now consider the mapping: $(a_1, \cdots, a_k) \to (a_1 - m, \cdots, a_k - m)$

Clearly, this mapping establishes one to one correspondence between the following sets:

- The set of all least r-modulo-m partitions of n with exactly k parts and $a_k > r$.
- The set of all least r-modulo-m partitions of $n - mk$.

Note that, the cardinality of the latter set is $R_k(n-mk,r,m)$.

Case(ii). Assume that $a_k = r$. Define $p_k(n,r)$ to be the number of partitions of n with exactly k parts and least part being equal to r.

Subcase (a). Assume that $a_{k-1} = r$.

In this case, the mapping $(a_1, \cdots, a_{k-1}, a_k) \to (a_1, \cdots, a_{k-1})$ establishes one to one correspondence between the following sets:

- The set of all least r-modulo-m partitions of n with $a_k = r$ and $a_{k-1} = r$.
- The set of all partitions of $n - r$ with exactly $k-1$ parts and having its least part equal to r.

Note that, the cardinality of the latter set is $p_{k-1}(n-r,r)$.

Subcase (b). Assume that $a_{k-1} \ne r$. In this case, the mapping $(a_1, \cdots, a_{k-1}, a_k) \to (a_1 - r, \cdots, a_{k-1} - r)$ establishes one to one correspondence between the following sets:

- The set of all least r-modulo-m partitions of n with $a_k = r$ and $a_{k-1} \neq r$.
- The set of all partitions of $n - kr$ with exactly $k-1$ parts.

Note that, the cardinality of the latter set is $p(n-kr, k-1)$.

Accordingly, we have

$$p_k(n,r) = p_{k-1}(n-r,r) + p(n-kr, k-1)$$

Applying this identity $k-1$ times, we get

$$p_k(n,r) = \sum_{i=1}^{k-1} p(n-kr, i)$$

Thus, the lemma 2.1 follows.

A characteristic found in the function $p(n,k)$ namely quasi polynomial representation is highly impetus for the further development of this subject.

Lemma 2.2. [4] The partition function, $p(n,k)$, is a quasi polynomial with quasi period $k!$ and each constituent polynomial $p(k!\,l + r, k)$ is of degree $k-1$ with leading coefficient being $\frac{k!^{k-2}}{(k-1)!}$, where $0 \leq r \leq k! - 1$.

As a corollary of Lemma 2.1 we find the generating function of $R_k(n,r,m)$.

Theorem 2.3. We have

$$\sum_{n=0}^{\infty} R_k(n,r,m)x^n = \frac{x^{rk}}{1-x^{km}}\left(\sum_{i=1}^{k-1} \frac{x^i}{(1-x)(1-x^2)\cdots(1-x^i)}\right)$$

for $|x| < 1$, $x \in R$.

Proof: It is known that the generating function of $p(n,k)$ is: $\sum_{n \geq k} p(n,k)\, x^n = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}$ when $|x| < 1$, $x \in R$.

Define $G(x) = \sum_{n=0}^{\infty} R_k(n,r,m)x^n$

Then as an application of Lemma 2.1, we get

$$\sum_{n=0}^{\infty}(R_k(n,r,m) - R_k(n-km,r,m))\, x^n = \sum_{n=1}^{\infty}\sum_{i=1}^{k-1} p(n-rk, i)\, x^n$$

This gives $(1-x^{km})G(x) = x^{rk}\left(\sum_{i=1}^{k-1} \frac{x^i}{(1-x)(1-x^2)\cdots(1-x^i)}\right)$ as expected.

3. Asymptotic estimate of $R_k(n,r,m)$

In this section, we derive an asymptotic estimate for the function $R_k(n,r,m)$.

Theorem 3.1. We have $R_k(n,r,m) \sim \frac{1}{m}\frac{n^{k-1}}{k!(k-1)!}$

Proof. Let $s \in \{0,1,\cdots,(mk)!-1\}$. Applying lemma 2.1 $(mk-1)!$ times, we get

$$R_k((mk)!\,l + s, r, m) - R_k((mk)!\,(l-1) + s, r, m) = \sum_{i=1}^{k-1}\sum_{j=0}^{(mk-1)!-1} p((mk)!\,l + s - rk - (mk)j, i) \tag{3.1}$$

Now, we see that, for every $i = 1,2,\cdots,k-1$ and $j = 0,1,\cdots,(mk-1)! - 1$ $p((mk)!\,l + s - rk - (mk)j, i) = p\left(i!\left(\frac{(mk)!}{i!}l + q_j\right) + r_j, i\right)$

where q_j and r_j were determined from the equality: $s - rk - (mk)j = (i!)q_j + r_j$. Uniqueness of q_j and r_j and the bound $0 \leq r_j \leq i! - 1$ follows as a consequence of Euclid's division algorithm.

Then, from Lemma 2.2, it follows that $p\left(i!\left(\frac{(mk)!}{i!}l + q_j\right) + r_j, i\right)$ is a polynomial in l of degree $i-1$ with leading coefficient being $\frac{i!^{i-2}(mk)!^{i-1}}{(i-1)!\,i!^{i-1}}$ for every $i = 1,2,\cdots,k-1$ and $j = 0,1,\cdots,(mk-1)! - 1$.

In particular,

$$p\left((k-1)!\left(\frac{(mk)!}{(k-1)!}l + q_j\right) + r_j, k-1\right)$$

is a polynomial in l of degree $k-2$ with leading coefficient being $\frac{(mk)!^{k-2}}{(k-2)!(k-1)!}$. Replacing l by $1,2,\cdots,l$ in eqn (3.1) and adding, one can get $R_k((mk)!\,l + s, r, m)$ as a polynomial of degree $k-1$ for every $s = 0,1,\cdots,(mk)! - 1$.

Now, we calculate the leading coefficient of $R_k((mk)!\,l + s, r, m)$. Let c_{k-1} be the leading coefficient of $R_k((mk)!\,l + s, r, m)$. Then from the previous observations, we have $(k-1)c_{k-1} = \frac{(mk)!^{k-2}(mk-1)!}{(k-2)!(k-1)!}$

This gives $c_{k-1} = \frac{(mk)!^{k-2}(mk-1)!}{(k-1)!^2}$

This yields the following limit $\lim_{l\to\infty} \frac{R_k((mk)!\,l + s, r, m)}{((mk)!\,l + s)^{k-1}} = \frac{1}{m}\frac{1}{k!(k-1)!}$

Since the above limit is valid for every $s \in \{0,1,\cdots,(mk)! - 1\}$,

we get $R_k(n,r,m) \sim \frac{1}{m}\frac{n^{k-1}}{k!(k-1)!}$

as desired.

Remark 3.2. From the estimates of the function $p(n,k)$(see [4]) and $R_k(n,r,m)$, it follows that: $\frac{R_k(n,r,m)}{p(n,k)}$ tends to $\frac{1}{m}$ as n tends to infinity. In other sense, if n is very large, then the probability that a partition of n with exactly k parts to be a least r-modulo-m partition is nearly $\frac{1}{m}$.

4. Ramanujan type congruences

Ramanujan discovered various surprising congruences for $p(n)$ when n is in certain special arithmetic progression; for example

$p(5n+4) \equiv 0 (mod\ 5)$ $p(7n+5) \equiv 0 (mod\ 7)$ and $p(11n+6) \equiv 0 (mod\ 11)$

There are now many proofs of these congruences (and their generalizations) in the literature (see [1]-[3] and [5]). We term such sort of congruences as Ramanujan Type congruences. We find Ramanujan type congruences of the function $R_k(n,r,m)$ by making use of the fact that it is a quasi polynomial. As we see, this fact is established in the course of proof of Theorem 3.1.

Following lemmas form crucial part of this section.

Lemma 4.1. Let $f(x)$ be a polynomial of degree k with rational coefficients such that $f(i)$ is an integer for every integer $i \geq 0$ and let p be a prime number greater than k. If r is a non negative integer such that $f(r) \equiv 0 (mod\ p)$ then, we have $f(np + r) \equiv 0 (mod\ p)$ for every positive integer n.

Proof. In the light of Newton's forward-difference formula, we can write $f(n) = \binom{n-1}{0}d_1 + \binom{n-1}{1}d_2 + \cdots + \binom{n-1}{k}d_{k+1}$
for every positive integer n,
where
$d_i = \Delta^i(f(1)) = \binom{i-1}{0}f(i) - \binom{i-1}{1}f(i-1) + \cdots + (-1)^{i-1}\binom{i-1}{i-1}f(1)$
for every $i = 1,2,\cdots,k+1$, where Δ^i denote the i^{th} iteration of forward difference operator.
Further, we notice that

$$\binom{k+m}{0}f(k+m+r) - \binom{k+m}{1}f(k+m+r-1) + \cdots + (-1)^{k+m}\binom{k+m}{k+m}f(r) = 0 \quad (4.1)$$

for every positive integer m, since $\Delta^{k+m}(f(r)) = 0$ for all positive integer m.
Let p be a prime number greater than k. Suppose that $f(r) \equiv 0(mod\ p)$
for some non negative integer r. Since $p > k$, we can write $p = k+m$ for some positive integer m. It is well known that
$\binom{k+m}{i} \equiv 0(mod\ (k+m))$
for $1 \le i \le k+m-1$.Thus from (4.1) it follows that $f(k+m+r) \equiv 0(mod\ (k+m))$,that is,$f(p+r) \equiv 0(mod\ p)$.Repeated application of this procedure for n times as before gives $f(np+r) \equiv 0(mod\ p)$.
Hence the lemma.

Lemma4.2. Let $f(x)$ be a polynomial with integer coefficients and let $m > 1$ be a positive integer. If r is a positive integer such that $f(r) \equiv 0(mod\ m)$then, we have$f(mn+r) \equiv 0(mod\ m)$for every positive integer n.

Proof. Let $f(x) = a_0 + a_1x + \cdots + a_kx^k$ be a polynomial of degree k with integer coefficients. Suppose that$f(r) \equiv 0(mod\ m)$
for some non negative integer r and positive integer $m > 1$. Then for any positive integer n, we have

$$f(mn+r) = f(r) + \sum_{i=1}^{k} a_i\left(\sum_{j=1}^{i}\binom{i}{j}(mn)^j r^{i-j}\right) \equiv 0(mod\ m)$$

Thus lemma follows.
In the following result we give Ramanujan type congruences for the function $R_3(n,0,2)$ as an application of lemma 4.1 and lemma 4.2. This method can be adopted for any other arithmetic function which is a quasi polynomial and integer valued.

Theorem 4.3. We have
(i) $R_3(1440n+728,0,2) \equiv 0(mod\ 2)$
(ii) $R_3(22320n+728,0,2) \equiv 0(mod\ 31)$
(iii) $R_3(3600n+13,0,2) \equiv 0(mod\ 5)$
(iv) $R_3(2160n+10,0,2) \equiv 0(mod\ 3)$
(v) $R_3(5040n+14,0,2) \equiv 0(mod\ 7)$

Proof. In the light of lemma 2.1, we can write $R_3(n,0,2) - R_3(n-6,0,2) = p(n,1) + p(n,2)$
Then taking $n = 6!\,l + s$ and application of lemma 2.1 for 5! times gives the following chain of equalities:

$$R_3(6!\,l+s,0,2) - R_3(6!\,(l-1)+s,0,2) = \sum_{i=1}^{5!} R_3(6!\,l-(i-1)6+s,0,2) - R_3(6!\,l-i6+s,0,2) = \sum_{i=1}^{5!} p(6!\,l-6(i-1)+s,1) + p(6!\,l-6(i-1)+s,2) =$$

$$\sum_{i=1}^{5!}\left(1+\left\lfloor\frac{6!\,l-6(i-1)+s}{2}\right\rfloor\right)$$

$$= 5! + \begin{cases}\displaystyle\sum_{i=1}^{5!}\frac{6!\,l-6(i-1)+s}{2} & \text{when } s \text{ is even}\\ \displaystyle\sum_{i=1}^{5!}\frac{6!\,l-6(i-1)+s-1}{2} & \text{when } s \text{ is odd}\end{cases} = 5! + \begin{cases}\frac{1}{2}\left(5!\,6!\,l - \frac{(5!-1)6!}{2} + 5!\,s\right) & \text{when } s \text{ is even}\\ \frac{1}{2}\left(5!\,6!\,l - \frac{(5!-1)6!}{2} + 5!\,(s-1)\right) & \text{when } s \text{ is odd}\end{cases}$$

Replacing l by $1,2,\cdots,l$ and adding we get

$$R_3(6!\,l+s,0,2) - R_3(s,0,2) = 5!\,l + \begin{cases}\frac{1}{2}\left(\frac{5!\,6!\,l(l+1)}{2} - \frac{(5!-1)6!}{2}l + 5!\,sl\right) & \text{when } s \text{ is even}\\ \frac{1}{2}\left(\frac{5!\,6!\,l(l+1)}{2} - \frac{(5!-1)6!}{2}l + 5!\,(s-1)l\right) & \text{when } s \text{ is odd}\end{cases}$$

Put $f_s(l) = R_3(6!\,l+s,0,2)$.
Then from the above expression for $f_s(l)$, we have $f_8(1) = 22382 \equiv 0(mod\ 2,31)$. Then, by lemma 4.2, we get

$f_8(2n+1) = R_3(6!\,(2n+1)+8,0,2) = R_3(1440n+728,0,2) \equiv 0(mod\ 2)$.
Similarly, we get $R_3(22320n+728,0,2) \equiv 0(mod\ 31)$.
Further we see that:
$f_{13}(0) \equiv 0(mod\ 5)$, $f_{10}(0) \equiv 0(mod\ 3)$and
$f_{14}(0) \equiv 0(mod\ 7)$.
Then applying lemma 4.1 we will get the remaining congruences.

Conclusion: Our endeavor in this article is to find asymptotic estimate and Ramanujan type congruences of a restricted partition function via a linear recurrence relation. This procedure seems to be yielding only when the function concerned is a quasi polynomial. This methodology can be applied to other functions which are not considered in this article provided they were quasi polynomials and meet the requisites prescribed in this article.

References:

1. G. Andrews, *The theory of Partitions*, Addison-Wesley, 1976.
2. G. Andrews and F. Garvan, *Dyson's crank of a partition*, Bull. Am. Math. Soc. 18(1988), 167-171.
3. A. O. L. Atkin, *Proof of a conjecture of Ramanujan*, Glasgow Math. J. 8(1967), 14-32.
4. A. David Christopher, M. Davamani Christober, *Estimate of five restricted partition functions that are quasi polynomial*, Bulletin of Mathematical Sciences, published online, doi:10.1007/s13373-014-0053-7
5. F. Garvan, A *simple proof of Watson's partition congruence for powers of 7*, J. Australian Math. Soc. (A) 36(1984), 316-334.
6. Emil Grosswald, *Some theorem concerning partitions*, Trans. Amer. Math. Soc., 89(1958), pp. 113-128.
7. G. H. Hardy and S. Ramanujan, *Asymptotic formula in Combinatorial analysis*, proc. Lond. Math. Soc. (2)17(1918), 75-115.
8. Iseki. S, *A partition function with some congruence condition*, Amer. J. Math. 81(1959), 939-961.
9. J. Lehner, *A partition function connected with the modulus five*, Duke Math. J. 8(1941), 631-655.
10. J. Livingood, *A partition function with prime modulus* $p \geq 3$, Amer. J. Math. 67(1945), 194-208.
11. Peter Hagis, *A problem on partitions with prime modulus* $p \geq 3$, Trans. Amer. Math. Soc., 102(1962), pp. 30-62.
12. Peter Hagis, *Partitions into unequal parts satisfying certain congruence conditions*, Journal of Number theory. 3, 115-123(1971).
13. Rademacher. H, *On the partition function* $p(n)$, Proc. London Math. Soc. (2) 43 (1937), 241-254.
14. M. M. Robertson and D. Spencer, *Partitions of large multi partites with congruence conditions,* Trans. Amer. Math. Soc. 219 (1976), pp. 299-322.
15. Subramanyasastri. V.V, *Partitions with congruence conditions*, J. Indian Math Soc. 36(1972), 177-194.

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