EMBEDDING THEOREM ON EDGE TOPOGENIC GRAPHS

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Abstract: This paper aims to construct an edge topogenic embedding H of a graph G. A graph G is said to be embedded in a graph H if there exists a sub graph G' of H such that G is isomorphic to G'.We call H a host graph of G. Important graph theoretic parameters such as domination number , matching number and covering number of G and H are compared

Keywords: covering number, domination number, embedding, matching number.

Introduction: The notion of topogenic graphs was introduced by B.D. Achariya , K.A. Jermina and Jisha Elizabeth joy[1]. This motivated us to give edge analog of this concept [2]. Let G = (V,E) be any graph and X be a ground set. A set edge indexer of G is an injective function f $:E(G)\to P(X)$ such that the induced vertex function $f^{\oplus}:V\to 2^X-\{\varphi\}$ defined by $f^{\oplus}(v)=\left\{\bigcup_{e_i\in E(G)} f(e_i): e_i \text{ is incident at v}\right\}$ and $f^{\oplus}(V)=\{f^{\oplus}(v):\ v\in V\}$ is also injective. An edge topogenic indexer is a set edge indexer of G such that $f(E)\bigcup f^{\oplus}(V)$ is a topology on X. A graph is said to be edge topogenic if it admits an edge topogenic set indexer. In this paper an embedding theorem on edge topogenic graph is proved. Some graph theoretic parameters of G and H are compared.

Definition : A graph G is said to be embedded in a graph H, if there exists a subgraph G' of H such that G is isomorphic to G'. We call H a host graph of G.

Theorem 1: Every graph can be embedded in an edge topogenic graph.

Proof : Let G = (V, E) be any graph. Let f be a set edge indexer of G with respect to X. If $f(E(G))\cup f^{\oplus}(V(G))$ is a topology on X, then the graph itself is its edge topogenic host graph. If not there arise two cases.

Case 1 : $X\in f^{\oplus}(V(G))$

Let K_1 be the number of open sets that are to be added to $f(E(G))\cup f^{\oplus}(V(G))$ so that it is a topology on X. Augment K_1 isolated vertices to G and join each of these vertices to a vertex in G with label X. Corresponding to K_1 newly added vertices, we have K_1 number of edges and these K_1 edges receive the required open sets to be included in $f(E(G))\cup f^{\oplus}(V(G))$ in an injective manner. The resulting graph K_1 is obviously an edge topogenic host graph of G.

Case 2 : $X\notin f^{\oplus}(V(G))$

Choose a vertex $v_o\in G$ such that $\left|f^{\oplus}(v_o)\right|$ = max $\left\{\left|f^{\oplus}(v)\right|/v\in V(G)\right\}$

Claim: $f^{\oplus}(v_o)\notin f(E(G))$

If $f^{\oplus}(v_o)=f(e)$ for some $e=v_iv_j$ (say), then $f^{\oplus}(v_o)\subseteq f^{\oplus}(v_i)$ and $f^{\oplus}(v_o)\subseteq f^{\oplus}(v_j)$

$f^{\oplus}(v_o)$ is of maximum cardinality.

$\Rightarrow f^{\oplus}(v_o)=f^{\oplus}(v_i)=f^{\oplus}(v_j)$

$\Rightarrow\Leftarrow$ ($\because$ $f^{\oplus}$ is injective)

$\therefore f^{\oplus}(v_o)\notin f(E(G))$ ______1

Let $|X|=k$ and $\left|f^{\oplus}(v_o)\right|=k_1$. Let $A_1,A_2,...,A_r$ be the subsets of X with cardinality k_1.

By 1, $A_1,A_2,...,A_r$ cannot be edge labels. But they can occur as vertex labels.

Subcase 1: Exactly one A_i, $1\le i\le r$ occurs as vertex label.

Let $f^{\oplus}(v_0)=A_1$.Let G' be a graph with V(G') =V(G)$\cup\{v_1,v_2,...v_r\}$ and E(G')=$E(G)\cup\{v_0v_i,1\le i\le r\}$ Label new edges as $f(v_0v_i)$ $=A_i$ for $1\le i\le r$. Clearly $f^{\oplus}(v_0)=X$ in G'.

By case 1, G' can be embedded in an edge topogenic graph H. Hence H is an edge topogenic host graph of G.

<u>Subcase 2</u>: Suppose more than one A_i occur as a vertex label.

Suppose $f^{\oplus}$ (v_1) $=A_1$ and $f^{\oplus}$ (v_2) $=A_2$ .Add a new vertex v_s and join it with v_1 and v_2 .Let

f (v_1v_s) $=A_1$ and f(v_2v_s) $=A_2$. Let G' be the resultant graph. $f^{\oplus}$(v_s) $=A_1 \cup A_2$.Note that $|A_1 \cup A_2| \succ$ K, proof follows from case 1.If not repeat the process given in sub case (ii) for G'.Since $|X|$ is finite, after a finite number of steps we will get a graph G' with $X \in f^{\oplus}(V(G'))$

By case 1, G' can be embedded in an edge topogenic host graph H. Hence H is an edge topogenic host graph of G.

Theorem 2 : Every graph can be embedded in a connected edge topogenic graph H.

Proof : By embedding theorem, every graph can be embedded in an edge topogenic graph H. Therefore it is enough if we prove that H is connected.

Case 1: G is connected.

In an edge topogenic embedding of G, we are adding k_1 vertices and join them to a vertex v_o in G for which $f^{\oplus}(v_o) = X$. Hence if G is connected, H is connected.

Case 2: G is not connected.

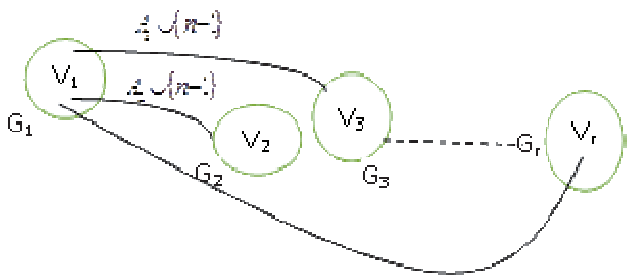
Let $G_1, G_2, \ldots, G_w$ be the components of G.

Let $v_1 \in G_1$ be such that $f^{\oplus}(v_1) = X$.

Let X = {1, 2, …n}, Y = {1, 2, …n, n+1}. For $2 \le i \le r$, choose a vertex v_i from G_i such that $|f^{\oplus}(v_i)|$ = $\max\{f^{\oplus}(v) : v \in G_i\}$.

Let A_i = $f^{\oplus}(v_i), 2 \le i \le r$. $A_1, A_2, \ldots A_r$ be the subsets of X with maximum cardinality and A_i = $|f^{\oplus}(v_i)|$, i = 2, 3,…,r

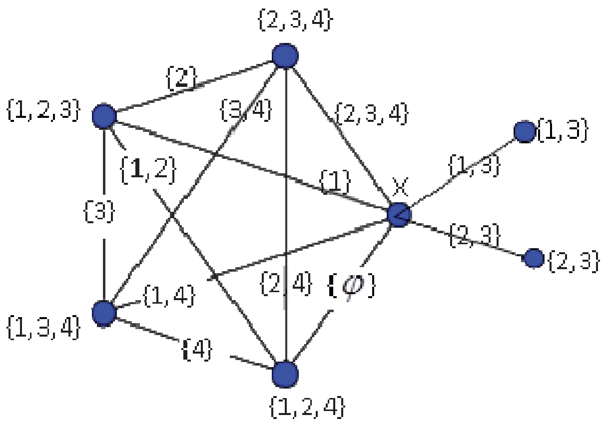
Now join v_1 with v_i $2 \le i \le r$ and extend f to E(G')by taking $f(v_1v_i) = A_i \cup \{n+1\}$, $2 \le i \le r$



In G', $f^{\oplus}(v_1) = Y$ and $f^{\oplus}(v_i) = A_i \cup \{n+1\}$, $2 \le i \le r$. The resulting graph G' is connected and by case 1, H is connected.

Hence H is a connected edge topogenic host graph of G.

Illustration: Connected embedding of K_5.



Theorem3: Let G be a graph with H as its edge topogenic host graph.

Then $\gamma(H) = \gamma(G)$ or $\gamma(H) = \gamma(G)+1$, where γ denotes the domination number.

Proof: If G is edge topogenic, then G = H and $\gamma(G) = \gamma(H)$

If G is not edge topogenic then $G \subset H$.

Case 1: X appears as a vertex label.

Let $v_0 \in V(G)$ such that $f^{\oplus}(v_0) = X$

In this case, by the construction of H described in embedding theorem, k vertices $v_1, v_2, \ldots, v_k$ together with k edges $v_0v_1, v_0v_2, \ldots, v_0v_k$ are added to G.

Let D be a minimal dominating set of G.

Subcase (i) : $v_0 \in D$

v_0 will take care of $v_1, v_2, \ldots, v_k$.

Hence, $\gamma(G) = \gamma(H)$

Subcase (ii) : $v_0 \notin D$

$v_1, v_2, \ldots, v_K$ being pendant vertices, v_0 is to be included in D.

Hence, $\gamma(H) = \gamma(G)+1$

Case 2: $X \notin \left\{ f^{\oplus}(v) \middle/ v \in V(G) \right\}$

Choose

v_0 *such that* $|f^{\oplus}(v_0)| = \max\{|f^{\oplus}(v)| (v \in V(G))\}$

Subcase (i) : If

$|f^{\oplus}(v_0)| > \{|f^{\oplus}(v)|, \forall v (\neq v_0) \in V(G)\}$

numbers of pendant edges incident at v_0 are added to G to get H.

So as in case 1, $\gamma(G) = \gamma(H)$ if $v_0 \in D$ and

$\gamma(H)=\gamma(G)+1$ if $v_0 \notin D$.

Subcase (ii): If $\left|f^{\oplus}(v_0)\right|=\left|f^{\oplus}(v_1)\right|$ for some $v_1(\neq v_0)\in V(G)$, then we move to a new vertex v, join to v_0, v_1 and label the edges with $f^{\oplus}(v_0)$ and $f^{\oplus}(v_1)$ respectively. Now in the new graph v will be the only vertex with $f^{\oplus}(v)$ as maximum of the vertex labels of the new graph.
Then v should be included in any minimal dominating set of H.

$\therefore\ \gamma(H)=\gamma(G)+1$

Hence, either $\gamma(H)=\gamma(G)$ or $\gamma(H)=\gamma(G)+1$

Theorem 4: Let G be a graph with H as its edge topogenic host graph. Then $\alpha'(H)=\alpha'(G)$ or $\alpha'(H)=\alpha'(G)+1$ where α' denotes the matching number of G.

Proof: If G is edge topogenic, then G = H and $\alpha'(H)=\alpha'(G)$

If G is not edge topogenic G $\subset$ H.

Case 1: X appears as a vertex label.

Let $v_0 \in V(G)$ such that $f^{\oplus}(v_0)=X$

In this case, by the construction of H described in embedding theorem, k vertices $v_1, v_2, \ldots, v_x$ together with k edges $v_0v_1, v_0v_2, \ldots, v_0v_k$ are added to G

Let M be a maximum matching in G.

Subcase (i) : v_0 is M- saturated.

Let uv$_0$ $\in M$. $M \cup \{v_0v_i\}$ is not a matching for $1 \le i \le k$ .Even if we remove uv$_o$ from M, and we can add at most one edge from the edges $\in M$ $v_0v_1, v_0v_2, \ldots, v_0v_k$ to M. Hence $\alpha'(H)=\alpha'(G)$.

Subcase (ii) : v_0 is M- unsaturated.

As v$_o$ is M-unsaturated, at most one edge from the pendant edges $v_0v_1, v_0v_2, \ldots, v_0v_k$ can be added to M. Hence, $\alpha'(K)=\alpha'(G)+1$

Case 2 : $X \notin \left\{ f^{\oplus}(v) \big/ v \in V(G) \right\}$

Choose

v_0 *such that* $\left|f^{\oplus}(v_0)\right|$

$=\max\left\{\left|f^{\oplus}(v)\right| \left(v \in V(G)\right)\right\}$

Subcase (i) :

$\left|f^{\oplus}(v_0)\right| > \left\{\left|f^{\oplus}(v)\right|, \forall v(\neq v_0)\left(\in V(G)\right)\right\}$ As in case 1, $\alpha'(G)=\alpha'(H)$ if v_0 is M- saturated and $\alpha'(H)=\alpha'(G)+1$ if v_0 is M-unsaturated.

Subcase (ii) : If $\left|f^{\oplus}(v_0)\right|=\left|f^{\oplus}(v_1)\right|$ for some $v_1(\neq v_0)\in V(G)$, then we move to a new vertex v, join to v_0, v_1 and label the edges with $f^{\oplus}(v_0)$ and $f^{\oplus}(v_1)$ respectively. Here both v_0, v_1 may be M-saturated or M- unsaturated or one of them is M-saturated and the other M-unsaturated .In all the cases at most one edge from the pendant edges $v_0v_1, v_0v_2, \ldots, v_0v_k$ can be added in the maximum matching M of G.

$\therefore\ \alpha'(H)=\alpha'(G)+1$

Hence, either $\alpha'(H)=\alpha'(G)$ or $\alpha'(H)=\alpha'(G)+1$

Theorem 5: Let G be a graph with H as its edge topogenic host graph. Then $\beta(H)=\beta(G)$ or $\beta(H)=\beta(G)+1$, where β denotes the covering number.

Proof: If G is edge topogenic, then G = H and $\beta(H)=\beta(G)$

If G is not edge topogenic G $\subset$ H.

Case 1 : X appears as a vertex label.

Let $v_0 \in V(G)$ such that $f^{\oplus}(v_0)=X$

In this case, by the construction of H described in embedding theorem, k vertices $v_1, v_2, \ldots, v_k$ together with k edges $v_0v_1, v_0v_2, \ldots, v_0v_k$ are added to G

Let S be a minimum covering of G.

Subcase (i) : $v_0 \in S$

As the pendant edges $v_0v_1, v_0v_2, \ldots, v_0v_k$ incident at v_0 , and $v_0 \in S$, $\beta(H)=\beta(G)$.

Subcase (ii) : $v_0 \notin S$

As $v_0v_1, v_0v_2, \ldots, v_0v_k$ being the pendant edges incident at v_0 , v_0 is to be included in S. Hence, $\beta(H)=\beta(G)+1$

Case 2 : $X \notin \left\{ f^{\oplus}(v) \middle/ v \in V(G) \right\}$

Subcase (i) :
Choose

v_0 *such that* $\left| f^{\oplus}(v_0) \right|$

$= \max \left\{ \left| f^{\oplus}(v) \right| \left(v \in V(G) \right) \right\}$

Number of pendant edges incident at v_0 are added to G to get H.So as in case 1, $\beta(H) = \beta(G)$ if $v_0 \in S$ and $\beta(H) = \beta(G) + 1$ if $v_0 \notin S$.

Subcase (ii) : If $\left| f^{\oplus}(v_0) \right| = \left| f^{\oplus}(v_1) \right|$ for some $v_1 (\neq v_0) \in V(G)$, then we move to a new vertex v, join to v_0, v_1 and label the edges with $f^{\oplus}(v_0)$ and $f^{\oplus}(v_1)$ respectively. Now in the new graph v will be the only vertex with $f^{\oplus}(v)$ as maximum of the vertex labels of the new graph G' Number of pendant edges incident at v are added to G' to get H.

Then v should be included in the minimum covering of H.

$\therefore \ \beta(H) = \beta(G) + 1$

Hence, either

$\beta(H) = \beta(G)$ or $\beta(H) = \beta(G) + 1$.

Conclusion: Construction of edge topogenic embedding may be refined so that some parameters of G and H are equal. If the refinement is not possible, edge topogenic index number may be determined for standard families of graphs.

References:

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