ON GAPS BETWEEN SUBSPACES OF A BANACH SPACE

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Abstract: The notion of gap between closed linear subspaces of a Banach space was introduced for the purpose of studying certain perturbation properties of closed linear operators. If *S* and *T* are (possibly unbounded) closed operators from a dense subspace of a Banach space *X* to a Banach space *Y*, their graphs *G(S)* and *G(T)* are closed linear subspaces of the Banach space $X \times Y$. Here, the gap between the operators *S* and *T* is defined as the gap between the closed spaces *G(S)* and *G(T)*. This notion is used to discuss certain limiting properties of closed operators as well.
In this paper, we discuss 'gap' between subspaces, which are not necessarily closed, and a limiting property of subspaces is discussed. Also, we discuss the gap convergence of the finite section method for unbounded closed operators.

Keywords: closed operators, filtrations, gap convergence, gap between subspaces.

Introduction: Notions such as convergence and approximation for bounded operators can be studied using norm. But, these notions cannot be discussed for unbounded operators using 'norm', as one cannot define a norm for unbounded operators. In this context, the notion of 'gap' seems to be useful to study the distance between unbounded closed operators and hence to study the convergence or approximation of unbounded operators to an extent.

1. Gap Between Subspaces

The gap between two subspaces of a Branch space is defined as follows :

1.1. Definition[3] If *X* and *Y* are two subspaces of a Banach space with $X \neq \{0\}$, we set

$$\delta(X,Y) = \sup_{u \in S_X} \operatorname{dist}(u,Y), \quad (1)$$

where $S_X = \{x \in X / \|x\| = 1\}$ is the unit sphere of *X*, and dist $(u,Y) = \inf\{\|u-y\| / y \in Y\}$ is the distance from u to *Y*. If $X = \{0\}$, we set $\delta(0, Y) = 0$. Now the gap between *X* and *Y* is defined as

$$\hat{\delta}(X, Y) = \max\{\delta(X,Y), \delta(Y,X)\}. \quad (2)$$

1.2. Remark: If $X \subseteq Y$, $\delta(X, Y) = 0$.

Hence, $\hat{\delta}(X, Y) = \delta(Y, X)$ if $X \subseteq Y$.

1.3. Theorem Let *Y* be a subspace of a Banach space *X*. Then $\hat{\delta}(X,Y) = 0$ if and only if *Y* is dense in *X*.

Proof:

Assume that *Y* is dense in *X*. As $Y \subseteq X$, $\delta(Y,X) = 0$.

Now let $u \in S_X$. That is, $u \in X$ and $\|u\| = 1$.

Since *Y* is dense in *X*, there exists a sequence $\{y_n\}$ in *Y* such that $\{y_n\}$ converges to *u*.

That is, $\|u - y_n\| \to 0$.

Hence, dist $(u, Y) = \inf\{\|u-y\| / y \in Y\} = 0$

Thus, for every $u \in S_X$, dist$(u, Y) = 0$.

So, $\sup_{u \in S_X} \operatorname{dist}(u,Y) = 0$.

That is, $\delta(X,Y) = 0$.

Now $\hat{\delta}(X,Y) = \max\{\delta(X,Y), \delta(Y,X)\} = 0$.

Conversely, assume that $\hat{\delta}(X,Y) = 0$.

That is, $\max\{\delta(X,Y), \delta(Y,X)\} = 0$.

Then, $\delta(X,Y) = 0$ and $\delta(Y,X) = 0$.

Take $\delta(Y,X) = 0$. That is, $\sup_{u \in S_X} \operatorname{dist}(u,Y) = 0$, which implies that dist$(u, Y) = 0$ for every $u \in S_X$.

That is, $\inf\{\|u-y\| / y \in Y\} = 0$ for every $u \in S_X$. So, for every $u \in S_X$, there is a sequence $\{y_n\}$ in *Y* such that (y_n) converges to *u*.Now, let $x \in X$.

If $x = 0$, then, since $0 \in Y$, $\{0, 0, 0, \ldots.\}$ is a sequence in *Y* converging to $x = 0$.

Suppose $x \neq 0$.

Put $z = \frac{x}{\|x\|}$. Then, $\|z\| = 1$ so that $z \in S_X$.

There exists a sequence $\{y_n\}$ in *Y* such that $y_n \to \frac{x}{\|x\|}$.

This implies that $\|x_n\| y_n \to x$ and $\{\|x_n\| y_n\}$ is a sequence in *Y*. Hence *Y* is dense in *X*.

1.4. Theorem Let *X* be a Banach space and (X_n) be a sequence of subspaces of *X* such that $X_n \subseteq X_{n+1}$ for every *n*. Let $Z = \bigcup_n X_n$.

Then, $\hat{\delta}(Z, X) = \lim_{n\to\infty} \hat{\delta}(X_n, X)$.

Proof: As $Z \subseteq X, \delta(Z, X) = 0$

so that $\hat{\delta}(Z, X) = \delta(X, Z)$.

Similarly, as $X_n \subseteq X, \delta(X_n, X) = 0$

and $\hat{\delta}(X_n, X) = \delta(X, X_n)$.

Hence, it is enough to prove that

$\lim_{n\to\infty}\delta(X,\ X_n) = \delta(X,Z)$.

Put $\delta_n = \delta(X, X_n)$ for every n and $\delta = \delta(X,Z)$.

We want to prove that $\delta_n \to \delta$.

Let $u \in S_X$.

Then, $\inf\{\|u-x\| / x \in X_n\} \geq \inf\{\|u-x\| / x\in Z\}$

since $X_n \subseteq Z$. (3)

Therefore,

$\sup_{u\in S_X}\inf\{\|u-x\| / x \in X_n\} \geq \sup_{u\in S_X}\inf\{\|u-x\| / x\in Z\}$

or, $\delta_n \geq \delta$. (4)

Let $\varepsilon > 0$ be given.

$\delta = \delta(X,Z) = \sup_{u\in S_X}\inf\{\|u-z\| / z\in Z\}$.

This implies that $\inf\{\|u-z\| / z\in Z\} \leq \delta$

for every $u \in S_X$.

So, there exists $x_0 \in Z$ such that $\|u - x_0\| < \delta + \varepsilon$.

Since $Z = \bigcup_n X_n$, $x_0 \in X_m$ for some m.

Then, $x_0 \in X_n$ for all $n \geq m$, since

$X_1 \subseteq X_2 \subseteq X_3 \subseteq .....$

So, $\inf\{\|u-x\| / x \in X_n\} < \delta + \varepsilon$ for all $n \geq m$.(5)

As (5) is true for every $u \in S_X$,

$\sup_{u\in S_X}\inf\{\|u-x\| / x \in X_n\} \leq \delta + \varepsilon$ for all $n \geq m$. (6)

That is $\delta_n \leq \delta + \varepsilon$ for all $n \geq m$.

Thus, for any $\varepsilon > 0$, there exists m such that $\delta \leq \delta_n \leq \delta + \varepsilon$ for all $n \geq m$.

This implies that $\delta_n \to \delta$. □

1.5. Corollary Let (X_n) be a sequence of subspaces of a Banach space X such that $X_1 \subseteq X_2 \subseteq X_3 \subseteq .....$

Then, $\hat{\delta}\left(\bigcup_n X_n, X\right) = \inf_n \hat{\delta}(X_n, X)$.

Proof: With the notations used in Theorem 1.4., it is enough to prove that $\delta = \inf_n \delta_n$

Suppose $n \leq m$. Then, $X_n \subseteq X_m$.

By the same arguments used in Theorem 1.4 we can show that $\delta_n \geq \delta_m$.

So, $\{\delta_n\}$ is a monotonic decreasing sequence of real numbers, which is bounded below by δ.

Hence $\lim_{n\to\infty}\delta_n = \inf_n \delta_n$. So, $\delta = \inf_n \delta_n$. □

1.6. Theorem: Let X be a Banach space and (X_n) be a sequence of subspaces of X such that $X_n \subseteq X_{n+1}$ for all n. Then, $\bigcup_n X_n$ is dense in X if and only if $\hat{\delta}(X_n, X) \to 0$.

Proof: Put $Z = \bigcup_n X_n$.

Then, Z is dense in X if and only if $\hat{\delta}(Z, X) = 0$, by Theorem 1.3.

By Theorem 1.4, $\hat{\delta}(X_n, X)$ converges to $\hat{\delta}(Z, X)$.

Hence $Z = \bigcup_n X_n$ is dense in X if and only if $\hat{\delta}(X_n, X)$ converges to 0.

2. **The Finite Section Method and Gap Convergence:** Filterations play a significant roll in approximating spectral values and finding approximate solutions of operator equations.

Throughout this section H denotes a separable Hilbert space.

2.1 Definition [1] A sequence (H_n) of finite dimensional subspaces of H is said to be a filtration of H if $H_n \subseteq H_{n+1}$ for every n and $\bigcup_n H_n$ is dense in H.

Every separable Hilbert Space has a countable orthonormal basis. (One can refer [4] or [5] for details). Let (e_n) be a countable orthonormal basis for H. Then, (H_n) is a filtration of H, where H_n = span $\{e_1, e_2, \ldots, e_n\}$.

The following result follows directly from Theorem 1.6.

2.2. Proposition: Let (H_n) be a filtration of H.

Then, $\hat{\delta}(H_n, H) \to 0$

Let A be a densely defined closed (possibly unbounded) operator in H. That is, A is a linear operator : $D(A) \to H$, where $D(A)$, the domain of A, is a dense subspace of H and the graph $G(A) = \{(x, Ax) / x \in D(A)\}$ of A, is a closed subspace of $H \times H$. Let (e_n) be an orthonormal basis for H contained in $D(A)$. Let P_n be the projection operator ($P_n^2 = P_n$ and P_n is self-adjoint) with $R(P_n) = H_n$. Put $A_n = P_n A P_n |_{H_n}$. That is, A_n is the operator $P_n A P_n$ restricted to H_n. A_n are bounded operators defined on the finite-dimensional subspaces H_n. Now the question before us is : when does $A_n \to A$ in the 'gap' sense, or, when does $\hat{\delta}(A_n, A) \to 0$?

Suppose A is a closed operator in H, so that $G(A)$ is a closed subspace of the Hilbert space $H \times H$. So, $G(A)$ itself is a Hilbert space.

As $A_n = P_n A P_n |_{H_n}$, $A_n(x) = A(x)$ for every $x \in H_n$. Also, $H_n \subseteq H_{n+1}$ for every n. Hence $G(A_n) \subseteq G(A_{n+1})$. So, by Theorem 1.6, $\hat{\delta}(G(A_n), G(A)) \to 0$ if and only if

$\bigcup_n G(A_n)$ is dense in $G(A)$. That is, $\hat{\delta}(A_n, A) \to 0$ if and only if $\bigcup_n G(A_n)$ is dense in $G(A)$. We have thus proved:

2.3. Theorem: Let A be a densely – defined closed operator in a separable Hilbert space H, whose domain contains an orthonormal basis (e_n). Let H_n = span $\{e_1, e_2, \ldots, e_n\}$ and $A_n = P_n A P_n|_{H_n}$, where for each n, P_n is the projection on H with range $R(P_n) = H_n$. Then, $A_n \to A$ in the 'gap' sense if and only if $\bigcup_n G(A_n)$ is dense in the graph of A.

The sequence (A_n) is called the finite section method for A. Thus the finite section method for an unbounded closed operator A converges to A in the 'gap' sense if and only if $\bigcup_n G(A_n)$ is dense in $G(A)$.

Conclusion : We see that 'gap' is an effective tool to study the convergence of unbounded operators. It can, thus, be used to discuss various problems in approximation methods for unbounded operators in general, and to approximate solutions of unbounded operator equations and spectral values, in particular.

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