# LEFT MULTIPLICATIVE GENERALIZED DERIVATIONS ON SEMI PRIME RINGS

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**Abstract:**Let $R$ be a ring. A map $F: R \to R$ is called a left multiplicative generalized derivation, if $F(xy) = g(x)y + xF(y)$ is fulfilled for all $x, y$ in R,where$g : R \to R$ is any map (not necessarily derivation or additive map). The main purpose of this paper is to study the following conditions:
1)$F([x,y]) = \pm (xy \pm yx)$
2) $F(x \circ y) = \pm(xy \pm yx)$
3)$[F(x),y] \pm [x, G(y)] = 0$, for all $x, y$ in some appropriate subset of R.

**Keywords:** Semiprime ring, Derivation, Generalized derivation, Multiplicative generalized derivation, Left multiplicative generalized derivation.

**Introduction:** Daif and Bell in[1] proved that if a semiprime ring $R$ admits a derivation $d$such that either $d([x,y]) \pm [x,y] = 0$, for all x, y $\in$ R, then R is necessarily commutative.Hongan in[2] generalized the above mentioned result considering $R$ satisfying the conditions $d([x,y]) \pm [x,y] = 0$ inZ(R) ,for all $x, y \in I$. Later Ali et .al [3] explored the commutativity of a prime ring admitting a generalized derivation $F$ satisfying anyone of the following conditions:
$F([x,y]) \pm [x,y]$ in $Z(R)$
$F(x \circ y) \pm (x \circ y)$in $Z(R)$, for all $x, y$ in some appropriate subset of $R$.
Motivated by the work of Daif and Bell [1] we consider similar condition for the multiplicative generalized derivative on semi-prime ring $R$.Ali et. al[4] multiplicative generalized derivation $F$ satisfying the following conditions:
1) $F([x,y]) = \pm (xy \pm yx)$
2) $F(x \circ y) = \pm (xy \pm yx)$
3) $[F(x),y] \pm [x, G(y)] = 0$,for all $x, y \in I$.
In this paper we prove the above conditions by applying the left multiplicative generalized derivations on semiprime rings.
**Preliminaries:** Throughout this paper $R$ will denote an associative ring with centre$Z(R)$.For any $x, y$ in $R$ the symbol$[x, y] = xy - yx$ is called commutator and $(x \circ y) = xy + yx$ is called anti-commutator. We recall that R is semiprime if for any a in R,$aRa = 0$ implies $a = 0$. A center $Z(R)$ is defined as $Z(R) = \{z \in R \backslash [z,R] = o\}$ . An additive map $d: R \to R$ is called a derivation of $R$ if $d(xy) = d(x)y + xd(y)$ for all $x, y$ in R. An additive map $F: R \to R$is called a generalized derivation if there exists a derivation$d: R \to R$such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.Amap $F: R \to R$ (not necessarily additive) is called a multiplicative generalized derivation if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, where d is any map (not necessarily derivation or additive map). A map $F: R \to R$ (not necessarily additive) is called left multiplicative generalized derivation if $F(xy) = d(x)y + xF(y)$forall $x, y \in R$ , where $d$ is any map (not necessarily derivation or additive map).
We use the following basic identities:
$[xy, z] = x[y,z] + [x,z]y$ and
$[x, yz] = [x,y]z + y[x,z]$ for all $x, y, z \in R$.
$((xy) \circ z) = x(y \circ z) - [x,z]y = (x \circ z)y + x[y,z]$and
$(x \circ (yz)) = (x \circ y)z - y[x,z] = y(x \circ z) + [x,y]z$.

**Main results:**

**Lemma 1:** If $R$ is a semiprime ring and $I$ is an ideal of $R$, then $I$ is semiprime ring.

**Theorem 1:** If $R$ be a semiprime ring, and $I$ a nonzero ideal of$R$. Let $F : R \to R$ be a left multiplicative generalized derivation associated with the map $g : R \to R$ such that $F([x,y]) = \pm (xy \pm yx)$ for all $x, y \in I$. Then $[g(x), x] = 0$ , for all $x \in I$.

**Proof:**By the assumption we have
$F([x,y]) = \pm (xy \pm yx)$ forall $x, y \in I$. (1)
We replace$x$ by $yx$ in equation (1)
we get$F[yx, y] = \pm (yxy \pm yyx)$
$F([y,y]x + y[x,y]) = \pm y(xy \pm yx)$
$F(y[x,y]) = \pm y(xy \pm yx)$
$g(y)[x,y] + yF[x,y] = \pm y(xy \pm yx)$
From equation (1) we get
$g(y)[x,y] = 0$ for all $x, y \in I$ .(2)
We replace$x$ by $x\,g(y)$ in equation (2),
we get,
$g(y)[x\,g(y), y] = 0$
$g(y)([x,y]g(y) + x[g(y), y]) = 0$
$g(y)[x,y]g(y) + g(y)x[g(y), y] = 0$
From equation (2) we get
$g(y)x[g(y), y] = 0$
Which is further implies from lemma (1), Iis semiprime ring. We conclude that$[g(y), y] = 0\ \forall y \in I$.
Hence$[g(x), x] = 0$ for all $x \in I$. ■

**Theorem 2:** If $R$ be a semiprime ring, and $I$a nonzeroideal of R. Let $F : R \to R$ be left multiplicative generalized derivation associated with the map $g : R \to R$ such that$F(x \circ y) = \pm (xy \pm yx)$ for all $x, y \in I$.Then $[g(x), x] = 0$, for all $x \in I$.

**Proof:** By the assumption we have
$F(x \circ y) = \pm (xy \pm yx)$, forall $x, y \in I$. (3)
We replace$x$ by $yx$ in equation (3) we get
$F(yx \circ y) = \pm (yxy \pm yyx)$
$F((y \circ y)x + y(x \circ y)) = \pm y(xy \pm yx)$
$F(y(x \circ y)) = \pm y(xy \pm yx)$
$g(y)(x \circ y) + yF(x \circ y) = \pm y(xy \pm yx)$
From equation (3) the above expression we get
$g(y)(x \circ y) = 0$ for all $x, y \in I$. (4)

We replace $x$ by $xg(y)$in equation (4), we get,
$g(y)(x\,g(y) \circ y) = 0$
$g(y)(\,(x \circ y)g(y) + x\,[g(y), y]) = 0$
$g(y)(x \circ y)\,g(y) + g(y)\,x\,[g(y), y] = 0$
From equation (4) we get
$g(y)\,x\,[g(y), y] = 0$ which further implies that $[g(y), y]\,x\,[g(y), y] = 0$.
Which is further implies from lemma (1), $I$is semiprime ring. We conclude that $[g\,(y), y] = 0\ \forall\ y \in I$.
Hence$[g(x), x] = 0$ ,for all $x \in I$.

**Theorem 3:**If $R$ be a semiprime ring and $I$a non zero ideal of$R$. Let $F, G : R \to R$ be left multiplicative generalized derivation associated with the map $d, g : R \to R$ such that $[F(x), y] \pm [x, G(y)] = 0$ for all $x, y \in I$.Then $[d(x), x] = 0$ and $[g(x), x] =0$ ,for all$x \in I$.

**Proof:**i) Assume that
$[F(x), y] - [x\,G(y)] = 0$ ,
for all $x, y \in I$. (5)
We replace $y$ by $xy$ in equation (5) we get
$[F(x), xy] - [x, G(xy)] = 0$ for all
$x, y \in I[F(x), x]y + x\,[F(x), y] - [x, g(x)y + x\,G(y)] = 0$ $[F(x), x]y + x\,[F(x), y] - [x, g(x)]y - g(x)[x, y] - [x, x]G(y) - x\,[x, G(y)] = 0$
$x\,([F(x), y] - [x, G(y)]) + [F(x), x]y - g(x)\,[x, y] - [x, g(x)]y = 0$
From equation (5) we get
$[F(x), x]y - g(x)\,[x, y] - [x, g(x)]y = 0$ for all $x, y \in I$ . (6)
We replace $y$ by $y\,g(x)$ in equation
$[F(x), x]y\,g(x) - g(x)[x, y\,g(x)] - [x, g(x)]y\,g(x) = 0[F(x), x]y\,g(x) - g(x)[x, y]g(x) - g(x)y\,[x, g(x)] - [x, g(x)]y\,g(x) = 0$
$([F(x), x]y - g(x)[x, y] - [x, g(x)]y)g(x) - g(x)y\,[x, g(x)] = 0$ . (7)
From equation (6) we get
$g(x)y\,[x, g(x)] = 0$
Further implies $[x, g(x)]\,y\,[x, g(x)] = 0$
From lemma (1), we get $[x, g(x)] = 0$.
Now we replace$x$ by $yx$ in equation (5), we get
$[F(yx), y] - [yx, G(y)] = 0$
$[d(y)x + y\,F(x), y] - [yx, G(y)] = 0$
$d(y)\,[x, y] + [d(y), y]\,x + [y, y]F(x) + y\,[F(x), y] - [y, G(y)]\,x - y[x, G(y)] = 0$
$y\,\big([\,F(x), y] - [x, G(y)]\big) + d(y)[x, y] + [d(y), y]x - [y, G(y)]x =0$
From (5) we get
$d(y)[x, y] + [d(y), y]x - [y, G(y)]x = 0$
for all $x, y \in I$. (8)
We replace$x\ by\ xdy$ in equation (8) we get
$d(y)[xd(y), y] + [d(y), y]x\,d(y) - [y, G(y)]x\,d(y) = 0$
$d(y)[x, y]d(y) + d(y)\,x\,[d(y), y] + [d(y), y]\,x\,d(y) - [y, G(y)]x\,d(y) = 0$
$\big(d(y)[x, y] + [d(y), y]x - [y, G(y)]x\big)d(y) + d(y)x\,[d(y), y] = 0.$ (9)
From (8) and (9) we get
$d(y)x\,[d(y), y] = 0$
Further implies $[d(y), y] = 0$ for all $y \in I$.
That implies $[d(x), x] = 0$ for all $x \in I$.
ii) Assume that
$[F(x), y] + [x, G(y)] = 0$
for all $x, y \in I$. (10)
We replace$y$ by $xy$ in equation (5) we get
$[F(x), xy] + [x, G(xy)] = 0$
for all $x, y \in I$
$[F(x), x]y + x\,[F(x), y] + [x, g(x)\,y + x\,G(y)] = 0$
$[F(x), x]y + x\,[F(x), y] + [x, g(x)]\,y + g(x)[x, y] + [x, x]\,G(y) + x\,[x, G(y)] = 0$
$x\,([F(x), y] + [x, G(y)]) + [F(x), x]y + g(x)[x, y] + [x, g(x)] = 0$
From equation (1) we get
$[F(x), x]y + g(x)\,[x, y] + [x, g(x)]y = 0$
for all x, y ∈ I. (11)
We replace$y$ by $y\,g(x)$ in equation (11)
$[F(x), x]y\,g(x) + g(x)\,[x, y\,g(x)] + [x, g(x)]y\,g(x) = 0$
$[F(x), x]y\,g(x) + g(x)\,[x, y]\,g(x) + g(x)\,y\,[x, g(x)] + [x, g(x)]y\,g(x) = 0$
$([F(x), x]y + g(x)[x, y] + [x, g(x)]y)g(x) + g(x)y\,[x, g(x)] = 0$ (12)
From equation (11) and equation (12), we get,
$g(x)y\,[x, g(x)] = 0$
Further implies $[x, g(x)]\,y\,[x, g(x)] = 0$
From lemma (1), we get $[x, g(x)] = 0$
Now we replace$x$ by $yx$ in equation (5) we get
$[F(yx), y] + [yx, G(y)] = 0$
$[d(y)x + y\,F(x), y] + [yx, G(y)] = 0$
$d(y)[x, y] + [d(y), y]x + [y, y]F(x) + y\,[F(x), y] + [y, G(y)]x + y[x, G(y)] = 0$
$y\,([\,F(x), y] + [x, G(y)]) + d(y)[x, y] + [d(y), y]x + [y, G(y)]x = 0$
$d(y)[x, y] + [d(y), y]x + [y, G(y)]x = 0$
for all x, y ∈ I. (13)
We replace$x$ by $xdy$ in equation (13) we get
$d(y)[xd(y), y] + [d(y), y]x\,d(y) + [y, G(y)]x\,d(y) = 0$
$d(y)[x, y]d(y) + d(y)\,x\,[d(y), y] + [d(y), y]\,x\,d(y) + [y, G(y)]x\,d(y) = 0$
$(d(y)[x, y] + [d(y), y]x + [y, G(y)]x)d(y) + d(y)x[d(y), y] = 0.$ (14)
From (13) and (14) we get
$d(y)x\,[d(y), y] = 0$
Further implies $[d(y), y] = 0$ for all $y \in I$.
That implies $[d(x), x] = 0$ for all $x \in I$.

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