A PARAMETER-UNIFORM NUMERICAL METHOD FOR A COUPLED SYSTEM OF TWO SINGULARLY PERTURBED LINEAR REACTION-DIFFUSION EQUATIONS WITH DISCONTINUOUS SOURCE TERM SUBJECT TO MIXED TYPE BOUNDARY CONDITIONS

MAHABUB BASHA PATHAN, SHANTHI VEMBU

Abstract: In this paper, a coupled system of two singularly perturbed linear reaction-diffusion second order ordinary differential equations with discontinuous source term subject to mixed type boundary conditions is considered. The leading term of each equation is multiplied by a distinct small positive parameter. A central difference scheme is used on a piecewise-uniform Shishkin mesh to solve the problem. The scheme is proved to be almost first order parameter-uniform convergence. Numerical results are presented, which are in good agreement with the theoretical results.

Keywords: Coupled system of singularly perturbed reaction-diffusion equations, Discontinuous source term, Mixed type boundary conditions, Shishkin mesh.

Introduction: Singular perturbation problems (SPPs) arise in various fields of Science and Engineering. Especially, singularly perturbed system of reaction-diffusion problems occur in the modeling of different physical phenomena, such as turbulent interaction of waves and currents, predator-prey population dynamics and investigation of diffusion processes complicated by chemical reactions in electro analytic chemistry [1]. For detailed study on singular perturbation problems, one can refer to the books mentioned in [2]-[4]. A numerical method for singularly perturbed coupled system of two second order ordinary differential equations with equal parameters having discontinuous source term subject to Dirichlet boundary conditions was studied in [5] and the same method is extended for two different parameters with some modifications in [6]. Motivated by the works given in [1]-[10], the aim of the present paper is to obtain parameter-uniform convergence for the following coupled system of two singularly perturbed linear reaction-diffusion second order ordinary differential equations with discontinuous source term subject to mixed type boundary conditions:

$$P_1\,\bar{y}(x) \equiv -\varepsilon_1 y_1''(x) + a_{11}(x)y_1(x) + a_{12}(x)y_2(x) = f_1(x), \qquad (1)$$

$$P_2\bar{y}(x) \equiv -\varepsilon_2 y_2''(x) + a_{21}(x)y_1(x) + a_{22}(x)y_2(x) = f_2(x), \qquad (2)$$

$x \in \Omega^- \cup \Omega^+$,

$$\left.\begin{aligned} B_{10}y_1(0) &\equiv \beta_{11}y_1(0) - \beta_{12}y_1'(0) = p,\\ B_{20}y_2(0) &\equiv \beta_{21}y_2(0) - \beta_{22}y_2'(0) = r,\\ B_{11}y_1(1) &\equiv \gamma_{11}y_1(1) + \gamma_{12}y_1'(1) = q,\\ B_{21}y_2(1) &\equiv \gamma_{21}y_2(1) + \gamma_{22}y_2'(1) = s, \end{aligned}\right\} \quad (3)$$

where $\varepsilon_1, \varepsilon_2$ are small parameters such that $0 < \varepsilon_1 \le \varepsilon_2 \le 1$.

Assume that

$$a_{12} \le 0 \quad \text{and} \quad a_{21} \le 0, \ \forall x \in \bar{\Omega}, \quad (4)$$

and for some constant α,

$$\left.\begin{aligned} &0 < \alpha < \min_{\bar{\Omega}}\{\alpha_1, \alpha_2\},\\ &\alpha_1 = \min_{x\in\bar{\Omega}}\{a_{11}(x) + a_{12}(x)\},\\ &\alpha_2 = \min_{x\in\bar{\Omega}}\{a_{21}(x) + a_{22}(x)\}. \end{aligned}\right\} \quad (5)$$

Consequently,

$$a_{11}(x) > |a_{12}(x)|, \quad a_{22}(x) > |a_{21}(x)|, \ \forall x \in \bar{\Omega}, \quad (6)$$

Here $\Omega = (0,1), \bar{\Omega} = [0,1]$, $\Omega^- = (0, d)$, $\Omega^+ = (d, 1)$, $d \in \Omega$, and $y_1, y_2 \in Y \equiv C^1(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$, $\bar{y} = (y_1, y_2)^T$.

Further, we assume that the source terms f_1, f_2 are sufficiently smooth on $\bar{\Omega}\backslash\{d\}$, and their derivatives have jump discontinuity at the same point. Let the jump at d in any function is given by

$[\omega](d) = \omega(d+) - \omega(d-)$.

The problem (1)-(3) can be written in vector form as

$$-E\bar{y}''(x) + A(x)\bar{y}(x) = \bar{f}(x),$$

$$\begin{pmatrix} B_{10}y_1(0) \\ B_{20}y_2(0) \end{pmatrix} = \begin{pmatrix} p \\ r \end{pmatrix}, \begin{pmatrix} B_{11}y_1(1) \\ B_{21}y_2(1) \end{pmatrix} = \begin{pmatrix} q \\ s \end{pmatrix},$$

where $E = diag(\varepsilon_1, \varepsilon_2)$,

$$A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} \quad \text{and}$$

$$\bar{f}(x) = \big(f_1(x), f_2(x)\big)^T.$$

Notations: Throughout this paper, C denotes a positive generic constant that is independent of the singular perturbation parameter ε and also of discretization parameter N which may not be same at each occurrence. We use the maximum norm to study the convergence of numerical solution to the exact solution and is defined by $\|y\|_D = \sup_{x\in D}|y(x)|$, where D is a closed and bounded subset in $\bar{\Omega}$.

The following results can be proved by following similar arguments considered in [5] & [6].

Theorem-1: The problem (1)-(3) has a solution $\bar{y} = (y_1, y_2)^T$ with $y_1, y_2 \in Y$.

Theorem-2 (Maximum Principle): Suppose that $\bar{y} = (y_1, y_2)^T, y_1, y_2 \in Y$ satisfies $B_{j0}y_j(0) \ge 0, B_{j1}y_j(1) \ge 0$, for $j = 1,2$,

$P_1\bar{y}(x) \ge 0, P_2\bar{y}(x) \ge 0$, $x \in \Omega^- \cup \Omega^+$ and $[\bar{y}'](d) \le \bar{0}$. Then $\bar{y}(x) \ge \bar{0}, \forall\, x \in \bar{\Omega}$.

Lemma-1 (Stability Result): Let $A(x)$ satisfy (4)-(6). If $\bar{y} = (y_1, y_2)^T$ is the solution of (1)-(3), then

$$\|\bar{y}\|_{\bar{\Omega}} \le \max\left\{\begin{matrix} \|B_{10}y_1(0)\|, \|B_{20}y_2(0)\|, \\ \|B_{11}y_1(1)\|, \|B_{21}y_2(1)\|, \\ \dfrac{1}{\alpha}\|\bar{f}\|_{\Omega^-\cup\Omega^+} \end{matrix}\right\}.$$

To derive the sharper bounds on the derivatives of the solution, we decompose the solution $\bar{y}$ into smooth and singular components as $\bar{y} = \bar{v} + \bar{w}$.
The smooth component $\bar{v}$ is given by

$$\left.\begin{array}{l} \bar{P}\bar{v}(x) = \bar{f}(x), x \in \Omega^- \cup \Omega^+, \\ \bar{v}(x) = A^{-1}(x)\bar{f}(x), x \in \{0, d^-, d^+, 1\}, \end{array}\right\} \quad (7)$$

and the singular component $\bar{w}$ is given by

$$\left.\begin{array}{l} \bar{P}\bar{w}(x) = 0, x \in \Omega^- \cup \Omega^+, \\ \bar{w}(x) = \bar{y}(x) - \bar{v}(x), x \in \{0,1\}, \\ [\bar{w}](d) = -[\bar{v}](d), \\ [\bar{w}'](d) = -[\bar{v}'](d). \end{array}\right\} \quad (8)$$

To define the bounds on the derivatives, the following layer functions are used:

$$\left.\begin{array}{l} L_{11}(x) = e^{-x\sqrt{\alpha/\varepsilon_1}} + e^{-(d-x)\sqrt{\alpha/\varepsilon_1}}, \\ L_{12}(x) = e^{-(x-d)\sqrt{\alpha/\varepsilon_1}} + e^{-(1-x)\sqrt{\alpha/\varepsilon_1}}, \\ L_{21}(x) = e^{-x\sqrt{\alpha/\varepsilon_2}} + e^{-(d-x)\sqrt{\alpha/\varepsilon_2}}, \\ L_{22}(x) = e^{-(x-d)\sqrt{\alpha/\varepsilon_2}} + e^{-(1-x)\sqrt{\alpha/\varepsilon_2}}. \end{array}\right\} (9)$$

Adopting the technique used in [7], the following lemmas can be proved.

Lemma-2: The smooth component $\bar{v}$ and its derivatives satisfy the bounds given by
$\|\bar{v}^{(k)}\| \leq C$, for $k = 0,1,2$, with
$\|\overline{v_1}'''\| \leq C\,\varepsilon_1^{-1/2}$, and
$\|\overline{v_2}'''\| \leq C\,\varepsilon_2^{-1/2}, \forall\, x \in \Omega^- \cup \Omega^+$.

Lemma-3: The singular component $\bar{w}$ and its derivatives satisfy the bounds given by

$$|w_1(x)| \leq \begin{cases} C\,L_{21}(x), & x \in \Omega^- \\ C\,L_{22}(x), & x \in \Omega^+, \end{cases}$$

$$|w_2(x)| \leq \begin{cases} C\,L_{21}(x), & x \in \Omega^- \\ C\,L_{22}(x), & x \in \Omega^+; \end{cases}$$

$$|w_1'(x)| \leq \begin{cases} C\,(\varepsilon_1^{-1/2}L_{11}(x) + \varepsilon_2^{-1/2}\,L_{21}(x)), x \in \Omega^- \\ C\,(\varepsilon_1^{-1/2}L_{12}(x) + \varepsilon_2^{-1/2}\,L_{22}(x)), x \in \Omega^+, \end{cases}$$

$$|w_2'(x)| \leq \begin{cases} C\,(\varepsilon_2^{-1/2}\,L_{21}(x)), x \in \Omega^- \\ C\,(\varepsilon_2^{-1/2}\,L_{22}(x)), x \in \Omega^+; \end{cases}$$

$$|w_1''(x)| \leq \begin{cases} C\,(\varepsilon_1^{-1}L_{11}(x) + \varepsilon_2^{-1}\,L_{21}(x)), & x \in \Omega^- \\ C\,(\varepsilon_1^{-1}L_{12}(x) + \varepsilon_2^{-1}\,L_{22}(x)), & x \in \Omega^+, \end{cases}$$

$$|w_2''(x)| \leq \begin{cases} C\,(\varepsilon_2^{-1}\,L_{21}(x)), x \in \Omega^- \\ C\,(\varepsilon_2^{-1}\,L_{22}(x)), x \in \Omega^+; \end{cases}$$

$$|w_1'''(x)| \leq \begin{cases} C\,(\varepsilon_1^{-3/2}L_{11}(x) + \varepsilon_2^{-3/2}\,L_{21}(x)), x \in \Omega^- \\ C\,(\varepsilon_1^{-3/2}L_{12}(x) + \varepsilon_2^{-3/2}\,L_{22}(x)), x \in \Omega^+, \end{cases}$$

$$|w_2'''(x)| \leq \begin{cases} C\,\varepsilon_2^{-1}(\varepsilon_1^{-1/2}L_{11}(x) + \varepsilon_2^{-1/2}\,L_{21}(x)), \\ \quad x \in \Omega^-, \\ C\,\varepsilon_2^{-1}(\varepsilon_1^{-1/2}L_{12}(x) + \varepsilon_2^{-1/2}\,L_{22}(x)), \\ \quad x \in \Omega^+. \end{cases}$$

Lemma-4: Let $\varepsilon_1 < \frac{\varepsilon_2}{4}$. Then the decomposition of the components w_1 and w_2 is given by
$w_1(x) = w_{1,\varepsilon_1}(x) + w_{1,\varepsilon_2}(x)$,
$w_2(x) = w_{2,\varepsilon_1}(x) + w_{2,\varepsilon_2}(x)$, where

$$|w''_{1,\varepsilon_1}(x)| \leq \begin{cases} C\,\varepsilon_1^{-1}L_{11}(x), & x \in \Omega^- \\ C\,\varepsilon_1^{-1}L_{12}(x), & x \in \Omega^+, \end{cases}$$

$$|w''_{1,\varepsilon_2}(x)| \leq \begin{cases} C\,\varepsilon_2^{-3/2}L_{21}(x), & x \in \Omega^- \\ C\,\varepsilon_2^{-3/2}L_{22}(x), & x \in \Omega^+; \end{cases}$$

$$|w''_{2,\varepsilon_1}(x)| \leq \begin{cases} C\,\varepsilon_2^{-1}L_{11}(x), & x \in \Omega^- \\ C\,\varepsilon_2^{-1}L_{12}(x), & x \in \Omega^+, \end{cases}$$

$$|w''_{2,\varepsilon_2}(x)| \leq \begin{cases} C\,\varepsilon_2^{-3/2}L_{21}(x), & x \in \Omega^- \\ C\,\varepsilon_2^{-3/2}L_{22}(x), & x \in \Omega^+. \end{cases}$$

2. Discretization of the problem

A piecewise uniform Shishkin mesh of N mesh intervals on $\Omega^- \cup \Omega^+$ is used with the transition parameters $\tau_{11}, \tau_{12}, \tau_{21}$ and τ_{22} defined as follows:

$$\tau_{11} = min\left\{\frac{d}{8}, \frac{\tau_{21}}{2}, 2\sqrt{\varepsilon_1/\alpha}\,\ln N\right\},$$

$$\tau_{12} = min\left\{\frac{(1-d)}{4}, \frac{\tau_{22}}{2}, 2\sqrt{\varepsilon_1/\alpha}\,\ln N\right\},$$

$\tau_{21} = min\left\{\frac{d}{4}, 2\sqrt{\varepsilon_2/\alpha}\,\ln N\right\}$, and

$$\tau_{22} = min\left\{\frac{(1-d)}{4}, 2\sqrt{\varepsilon_2/\alpha}\,\ln N\right\}.$$

The interior points of the mesh are denoted by

$$\Omega_\varepsilon^N = \left\{x_i : 1 \leq i \leq \frac{N}{2} - 1\right\} \cup \left\{x_i : \frac{N}{2} + 1 \leq i \leq N - 1\right\}.$$

Clearly, $x_{N/2} = d$ and
$\bar{\Omega}_\varepsilon^N = \{x_i : i = 0,1,2, \ldots\ldots, N\}$.

We divide $\bar{\Omega}^{-N}$ into five sub-intervals $[0, \tau_{11}], [\tau_{11}, \tau_{21}], [\tau_{21}, d - \tau_{21}]$, $[d - \tau_{21}, d - \tau_{11}]$ and $[d - \tau_{11}, d]$.
On the sub-intervals $[0, \tau_{11}], [\tau_{11}, \tau_{21}]$, $[d - \tau_{21}, d - \tau_{11}]$ and $[d - \tau_{11}, d]$, a uniform mesh with $\frac{N}{16}$ mesh intervals is placed, while the sub-interval $[\tau_{21}, d - \tau_{21}]$ has a uniform mesh with $\frac{N}{4}$ mesh intervals. Similarly, we divide $\bar{\Omega}^{+N}$ into five sub-intervals $[d, d + \tau_{12}], [d + \tau_{12}, d + \tau_{22}]$, $[d + \tau_{22}, 1 - \tau_{22}], [1 - \tau_{22}, 1 - \tau_{12}]$ and $[1 - \tau_{12}, 1]$. On the sub-intervals $[d, d + \tau_{12}], [d + \tau_{12}, d + \tau_{22}]$, $[1 - \tau_{22}, 1 - \tau_{12}]$ and $[1 - \tau_{12}, 1]$, a uniform mesh with $\frac{N}{16}$ mesh intervals is placed, while the sub-interval $[d + \tau_{22}, 1 - \tau_{22}]$ has a uniform mesh with $\frac{N}{4}$ mesh intervals.

On the piecewise uniform mesh $\bar{\Omega}_\varepsilon^N$, a standard centered finite difference operator is used. Then the fitted mesh method for (1)-(3) is:

$$\left.\begin{array}{l} P_1^N\bar{y}_i \equiv -\varepsilon_1\delta^2\,y_1(x_i) + a_{11}(x_i)y_1(x_i) \\ \quad +a_{12}(x_i)y_2(x_i) = f_1(x_i), \;\; \forall x \in \Omega_\varepsilon^N, \\ P_2^N\bar{y}_i \equiv -\varepsilon_2\delta^2\,y_2(x_i) + a_{21}(x_i)y_1(x_i) \\ \quad +a_{22}(x_i)y_2(x_i) = f_2(x_i), \;\; \forall x \in \Omega_\varepsilon^N, \end{array}\right\} (10)$$

$$\left.\begin{array}{l} B_{10}^N y_1(x_0) \equiv \beta_{11}y_1(x_0) - \beta_{12}D^+y_1(x_0) = p, \\ B_{20}^N y_2(x_0) \equiv \beta_{21}y_2(x_0) - \beta_{22}D^+y_2(x_0) = r, \\ B_{11}^N y_1(x_N) \equiv \gamma_{11}y_1(x_N) + \gamma_{12}D^-y_1(x_N) = q, \\ B_{21}^N y_2(x_N) \equiv \gamma_{21}y_1(x_N) + \gamma_{22}D^-y_2(x_N) = s, \end{array}\right\} (11)$$

where $\left.\begin{array}{l} P_1^N\bar{y}_i \\ P_2^N\bar{y}_i \end{array}\right\} \equiv \boldsymbol{P}^N\bar{y}_i$,

$$\delta^2\bar{y}(x_i) = \frac{D^+\bar{y}(x_i) - D^-\bar{y}(x_i)}{\left(\frac{h_i + h_{i+1}}{2}\right)},$$

$D^{+}\bar{y}(x_i)=\frac{\bar{y}(x_{i+1})-\bar{y}(x_i)}{h_{i+1}}$,

$D^{-}\bar{y}(x_i)=\frac{\bar{y}(x_i)-\bar{y}(x_{i-1})}{h_i}$,

$h_i=x_i-x_{i-1}$ and $h_{i+1}=x_{i+1}-x_i$.

At the point $x_{N/2}=d$, $\bar{f}(d)$ is given by

$$\bar{f}(d)=\frac{f_{j1}\left(d-h_{\frac{N}{2}}\right)+f_{j2}\left(d+h_{\left(\frac{N}{2}\right)+1}\right)}{2},\ j=1,2.$$

Analogous to maximum principle in continuous case, the following discrete maximum principle can be proved.

Lemma-5: (Discrete maximum principle) For any mesh function $\boldsymbol{\omega}(x_i)$, assume that $B_{j0}^{N}\omega_j(x_0)\geq 0, B_{j1}^{N}\omega_j(x_N)\geq 0, for\ j=1,2$ and $\boldsymbol{P^N}\boldsymbol{\omega}(x_i)\geq 0, \forall x_i\in\Omega_\varepsilon^N$, then $\boldsymbol{\omega}(x_i)\geq \mathbf{0}$, $\forall x_i\in\Omega_\varepsilon^N$.

3. Error Analysis

By the Taylor's series expansion on smooth and singular components, we have

$$\left|\varepsilon_k\left(\frac{d^2}{dx^2}-\delta^2\right)v_k(x_i)\right| \leq \begin{cases} C\,\varepsilon_k(x_{i+1}-x_{i-1})|v_k|_3 & (12)\\ C\,\varepsilon_k h^2|v_k|_4 & (13)\end{cases}$$

$x_{i+1}-x_i=x_i-x_{i-1}=h$,

and

$$\left|\varepsilon_k\left(\frac{d^2}{dx^2}-\delta^2\right)w_k(x_i)\right| \leq \begin{cases} C\,\varepsilon_k(x_{i+1}-x_{i-1})|w_k|_3 & (14)\\ C\,\varepsilon_k h^2|w_k|_4, & (15)\\ C\,\varepsilon_k \max\limits_{x\in[x_{i-1},x_{i+1}]}\left|w_k''(x_i)\right|, & (16)\end{cases}$$

$x_{i+1}-x_i=x_i-x_{i-1}=h$

where $k=1,2, i\neq\frac{N}{2}$,

$$|y_k|_j=\max\left|\frac{d^jy}{dx^j}\right|, \forall\, j\in N.$$

Note that the mesh is uniform if $\tau_{21}=\frac{d}{4}$, $\tau_{22}=\frac{(1-d)}{4}, \tau_{11}=\frac{d}{8}, \tau_{12}=\frac{(1-d)}{8}$ and then N^{-1} is very small w.r.t. ε_1 and ε_2.

Using (12) and bounds on the smooth components, we have

$$|(\boldsymbol{P}^N-\boldsymbol{P})\boldsymbol{v}(x_i)|\leq\frac{(h_i+h_{i+1})}{3}\begin{pmatrix}\varepsilon_1\left\|v_1'''\right\|_{\Omega^-\cup\Omega^+}\\ \varepsilon_2\left\|v_2'''\right\|_{\Omega^-\cup\Omega^+}\end{pmatrix}\leq C\begin{pmatrix}N^{-1}\\N^{-1}\end{pmatrix}.$$

The error estimates for the singular components for different sub-intervals are given by

(i) Let $x_i\in[\tau_{21}, d-\tau_{21}]\cup[d+\tau_{22}, 1-\tau_{22}]$. For $x_i\in\left[\tau_{21},\frac{d}{2}\right]$, bounds on the singular components using (16) & (9), we have

$$|((\boldsymbol{P}^N-\boldsymbol{P})\boldsymbol{w})_1(x_i)|\leq C\,\|B_{21}\|_{[x_{i-1},x_{i+1}]}=B_{21}(x_{i-1}).$$

Thus, $\|B_{21}\|_{[x_{i-1},x_{i+1}]}\leq 2e^{\left(-\tau_{21}+\frac{16\tau_{21}}{N}\right)\sqrt{\frac{\alpha}{\varepsilon_2}}}$ $\leq C\,N^{-1}$. Similar result can be proved when $x_i\in\left[\frac{d}{2}, d-\tau_{21}\right]$ and also for $x_i\in[d+\tau_{22}, 1-\tau_{22}]$. Hence, for $x_i\in[\tau_{21}, d-\tau_{21}]\cup[d+\tau_{22}, 1-\tau_{22}]$, we have

$$\begin{pmatrix}|((\boldsymbol{P}^N-\boldsymbol{P})\boldsymbol{w})_1(x_i)|\\|((\boldsymbol{P}^N-\boldsymbol{P})\boldsymbol{w})_2(x_i)|\end{pmatrix}\leq\begin{pmatrix}CN^{-1}\\CN^{-1}\end{pmatrix}.$$

(ii) Let $x_i\in(0,\tau_{11})\cup(d-\tau_{11}, d)\cup(d, d+\tau_{12})\cup(1-\tau_{12}, 1)$.

Using (14) and the bounds on the singular components together with the inequality $h_i+h_{i+1}\leq 32\sqrt{\frac{\varepsilon_1}{\alpha}}\,N^{-1}\ln N$, we get

$$\begin{pmatrix}|((\boldsymbol{P}^N-\boldsymbol{P})\boldsymbol{w})_1(x_i)|\\|((\boldsymbol{P}^N-\boldsymbol{P})\boldsymbol{w})_2(x_i)|\end{pmatrix}\leq\frac{(h_i+h_{i+1})}{3}\begin{pmatrix}\varepsilon_1\left\|w_1'''\right\|\\ \varepsilon_2\left\|w_2'''\right\|\end{pmatrix}\leq C\begin{pmatrix}N^{-1}\ln N\\N^{-1}\ln N\end{pmatrix}.$$

(iii) Let $x_i\in(\tau_{11}, \tau_{21})\cup(d-\tau_{21}, d-\tau_{11})\cup(d+\tau_{12}, d+\tau_{22})\cup(d-\tau_{22}, d-\tau_{12})$.

Assume that $\frac{\varepsilon_2}{4}\leq\varepsilon_1\leq\varepsilon_2$. Using (14) and the bounds on the singular components, we have

$$\begin{pmatrix}|((\boldsymbol{P}^N-\boldsymbol{P})\boldsymbol{w})_1(x_i)|\\|((\boldsymbol{P}^N-\boldsymbol{P})\boldsymbol{w})_2(x_i)|\end{pmatrix}\leq\frac{(h_i+h_{i+1})}{3}\begin{pmatrix}\varepsilon_1\left\|w_1'''\right\|\\ \varepsilon_2\left\|w_2'''\right\|\end{pmatrix}\leq C\begin{pmatrix}N^{-1}\ln N\\N^{-1}\ln N\end{pmatrix}.$$

If $\varepsilon_1<\frac{\varepsilon_2}{4}$, then we have

$$\begin{pmatrix}|((\boldsymbol{P}^N-\boldsymbol{P})\boldsymbol{w})_1(x_i)|\\|((\boldsymbol{P}^N-\boldsymbol{P})\boldsymbol{w})_2(x_i)|\end{pmatrix}\leq\begin{pmatrix}\varepsilon_1\left(D^2-\frac{d^2}{dx^2}\right)w_{1,\varepsilon_1}(x_i)\\ \varepsilon_2\left(D^2-\frac{d^2}{dx^2}\right)w_{2,\varepsilon_1}(x_i)\end{pmatrix}+\begin{pmatrix}\varepsilon_1\left(D^2-\frac{d^2}{dx^2}\right)w_{1,\varepsilon_2}(x_i)\\ \varepsilon_2\left(D^2-\frac{d^2}{dx^2}\right)w_{2,\varepsilon_2}(x_i)\end{pmatrix}.$$

Using the analysis in (i), it is easy to get

$$\begin{pmatrix}\varepsilon_1\left(D^2-\frac{d^2}{dx^2}\right)w_{1,\varepsilon_1}(x_i)\\ \varepsilon_2\left(D^2-\frac{d^2}{dx^2}\right)w_{2,\varepsilon_1}(x_i)\end{pmatrix}\leq 2\begin{pmatrix}\varepsilon_1\left\|w_{1,\varepsilon_1}'''\right\|_{[x_{i-1},x_{i+1}]}\\ \varepsilon_2\left\|w_{2,\varepsilon_1\varepsilon}'''\right\|_{[x_{i-1},x_{i+1}]}\end{pmatrix}\leq C\begin{pmatrix}N^{-1}\\N^{-1}\end{pmatrix}$$

and

$$\begin{pmatrix}\varepsilon_1\left(D^2-\frac{d^2}{dx^2}\right)w_{1,\varepsilon_2}(x_i)\\ \varepsilon_2\left(D^2-\frac{d^2}{dx^2}\right)w_{2,\varepsilon_2}(x_i)\end{pmatrix}\leq\frac{(h_i+h_{i+1})}{3}\begin{pmatrix}\varepsilon_1\left\|w_{1,\varepsilon_2}'''\right\|\\ \varepsilon_2\left\|w_{2,\varepsilon_2}'''\right\|\end{pmatrix}\leq C\begin{pmatrix}N^{-1}\ln N\\N^{-1}\ln N\end{pmatrix}.$$

Combining the results of (i)-(iii) for singular components, we get

$$\begin{pmatrix}|((\boldsymbol{P}^N-\boldsymbol{P})\boldsymbol{w})_1(x_i)|\\|((\boldsymbol{P}^N-\boldsymbol{P})\boldsymbol{w})_2(x_i)|\end{pmatrix}\leq C\begin{pmatrix}N^{-1}\ln N\\N^{-1}\ln N\end{pmatrix}.$$

At the point $x_{N/2}=d$,

let $h_{N/2}=h_{(N/2)+1}=h$. Then

$$((\boldsymbol{P}^N-\boldsymbol{P})\boldsymbol{w})_1(d)=\bar{f}_1(d)+\frac{\varepsilon_1}{h^2}\int_{t=d}^{d+h}\int_{s=d}^{t}y_1''(s)\,ds\,dt-\frac{\varepsilon_1}{h^2}\int_{t=d-h}^{d}\int_{s=d}^{t}y_1''(s)\,ds\,dt$$

$-a_{11}(d)y_1(d) - a_{12}(d)y_2(d)$

$$= \frac{1}{h^2}\int_{t=d}^{d+h}\int_{s=d}^{t}\int_{p=s}^{d+h} + \frac{1}{h^2}\int_{t=d-h}^{d}\int_{s=d}^{t}\int_{p=d-h}^{s}(f_1 - a_{11}y_1 - a_{12}y_2)'(p)\,dp\,ds\,dt$$

$-a_{11}(d)y_1(d) - a_{12}(d)y_2(d) + +\frac{1}{2}(a_{11}(d - h)y_1(d-h) + a_{12}(d-h)y_2(d-h))$
$+\frac{1}{2}(a_{11}(d+h)y_1(d+h) + a_{12}(d+h)y_2(d+h))$

$$= \frac{1}{h^2}\int_{t=d}^{d+h}\int_{s=d}^{t}\int_{p=s}^{d+h} + \frac{1}{h^2}\int_{t=d-h}^{d}\int_{s=d}^{t}\int_{p=d-h}^{s}(f_1 - a_{11}y_1 - a_{12}y_2)'(p)\,dp\,ds\,dt$$

$+\frac{1}{2}\int_{t=d}^{d-h}(a_{11}(t)y_1(t) + a_{12}(t)y_2(t))'dt$
$+\frac{1}{2}\int_{t=d}^{d+h}(a_{11}(t)y_1(t) + a_{12}(t)y_2(t))'dt,$

which gives
$((\boldsymbol{P}^N - \boldsymbol{P})\boldsymbol{w})_1(d) \leq C\,N^{-1}\ln N$ and
$((\boldsymbol{P}^N - \boldsymbol{P})\boldsymbol{w})_2(d) \leq C\,N^{-1}\ln N$.

Theorem-3: Let $\boldsymbol{y}$ and $\boldsymbol{Y}$ be the exact and the numerical solutions of the problem (1)-(3). Then for sufficiently large N, we get $\max_{x_i\in\Omega^N}|\boldsymbol{Y}(x_i) - \boldsymbol{y}(x_i)| \leq C\,N^{-1}\ln N$.

Proof: Consider the mesh function $\Psi(x_i) = C\,N^{-1}\ln N \pm (\boldsymbol{Y} - \boldsymbol{y})(x_i)$. It satisfies $B_{j0}^N\Psi_j(x_0) \geq 0, B_{j1}^N\Psi_j(x_N) \geq 0, for\ j = 1,2$ and $\boldsymbol{P^N\Psi}(x_i) \geq 0, \forall x_i \in \bar{\Omega}_\varepsilon^N$. Using discrete maximum principle, we get $\boldsymbol{\Psi}(x_i) \geq 0, \forall x_i \in \bar{\Omega}_\varepsilon^N$. Then the desired result follows.

4. Numerical Experiments

Example-1: Consider the following test problem:

$$-\varepsilon_1 y_1''(x) + 2(x+1)^2 y_1(x) - (1+x^3)y_2(x) = f_1(x), \quad x \in \Omega^- \cup \Omega^+,$$
$$-\varepsilon_2 y_2''(x) - 2\cos\left(\frac{\pi}{4}x\right)y_1(x) + 2.2\,e^{1-x}\,y_2(x) = f_2(x), \quad x \in \Omega^- \cup \Omega^+,$$
$$y_1(0) - y_1'(0) = 0, \quad 2y_1(1) + y_1'(1) = 1,$$
$$y_2(0) - 3y_2'(0) = 0, \quad y_2(1) + y_2'(1) = 1,$$

where $f_1(x) = \begin{cases} 2e^x, & 0 \leq x \leq 0.5 \\ 1, & 0.5 < x \leq 1 \end{cases}$ and

$$f_2(x) = \begin{cases} 10x+1, & 0 \leq x \leq 0.5 \\ 2, & 0.5 < x \leq 1. \end{cases}$$

Here we set $\alpha = 0.95$. As the exact solution of Example-1 is not known, we use double mesh principle. The double mesh differences are calculated by

$$D_{\varepsilon_1,\varepsilon_2}^N = \max_{x_j\in\bar{\Omega}_{\varepsilon_1,\varepsilon_2}^N}\left|\boldsymbol{Y}^N(x_j) - \boldsymbol{Y}^{2N}(x_j)\right|,$$

$j = 0,1,\dots,N$,
where $\boldsymbol{Y}^N(x_j)$ and $\boldsymbol{Y}^{2N}(x_j)$ denote the numerical solutions obtained using N and $2N$ mesh intervals respectively. If $\varepsilon_1 = 10^{-i}$ for some non-negative integer i, we take

$$D_{\varepsilon_1}^N = \max\left\{\begin{matrix} D_{\varepsilon_1,1}^N, D_{\varepsilon_1,10^{-1}}^N, D_{\varepsilon_1,10^{-2}}^N, \\ \dots, D_{\varepsilon_1,10^{-i}}^N \end{matrix}\right\}$$

and the parameter-uniform error is computed by $D^N = \max\{D_1^N, D_{10^{-1}}^N, D_{10^{-2}}^N, \dots, D_{10^{-16}}^N\}$. Then the order of convergence is obtained by using the formula $P^N = \log_2\left(\frac{D^N}{D^{2N}}\right)$.

Table-1: Maximum point-wise errors D^N and $\varepsilon_1, \varepsilon_2$ −uniform rate of convergence P^N for y_1 and y_2 of Example-1.

$0 < \varepsilon_1 \leq \varepsilon_2 \leq 1$	y_1		y_2	
	D^N	P^N	D^N	P^N
N=64	0.316470	**0.35**	0.662600	**0.29**
N=128	0.248620	**0.49**	0.542640	**0.47**
N=256	0.176680	**0.63**	0.391400	**0.63**
N=512	0.114140	**0.73**	0.253330	**0.74**
N=1024	0.058631	**0.80**	0.152180	**0.80**
N=2048	0.039375	**0.82**	0.087229	**0.81**

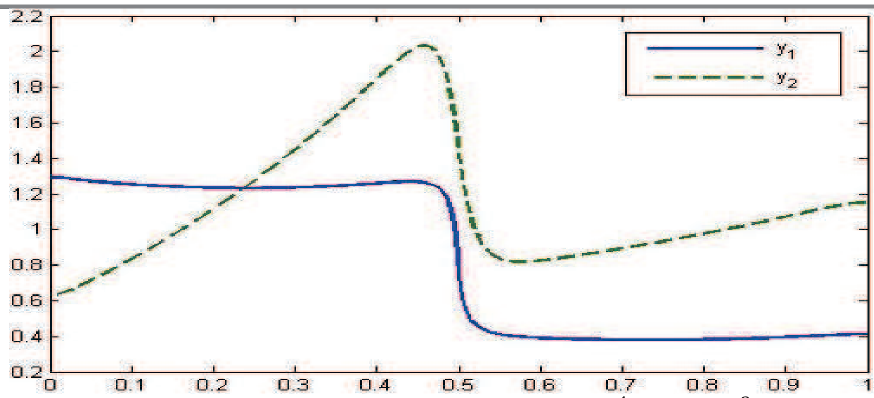

Figure-1: *Numerical solution of Example-1 for* $\varepsilon_1 = 10^{-4}, \varepsilon_2 = 10^{-3}$ *and* $N = 512$.

Conclusion: A coupled system of two singularly perturbed linear reaction-diffusion second order ordinary differential equations with discontinuous source term subject to mixed type boundary conditions was examined. A difference scheme using fitted mesh method on the Shishkin mesh was constructed for solving the problem. The obtained numerical results show almost first order of convergence which are supported by the theoretical results.

Acknowledgment The authors express their sincere thanks to the referees for their valuable comments.

References:

1. S. Chandra Sekhara Rao, S. Kumar, M. Kumar: Uniform global convergence of a hybrid scheme for singularly perturbed reaction-diffusion systems, J. Optim. Theory Appl., 2011, 151:338-352.
2. P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O'Riordan, G.I. Shishkin: Robust Computational Techniques for Boundary Layers, Chapman and Hall/CRC, Boca Raton, Florida,2000.
3. J.J.H. Miller, E. O'Riordan, G.I. Shishkin.: Fitted numerical methods for singular perturbation problems, World Scientific, Singapore, 1996.
4. R.E. O'Malley: Singular perturbation methods for ordinary differential equations, Springer Verlog, New York, 1991.
5. Tamilselvan, N. Ramanujam, V. Shanthi: A numerical method for singularly perturbed weakly coupled system of two second order ordinary differential equations with discontinuous source term, Journal of Computational and Applied Mathematics, 202, 2007, pp. 203-216.
6. S. Chandra Sekhara Rao, S. Chawla: Interior layers in coupled system of two singularly perturbed reaction-diffusion equations with discontinuous source term, in: I. Dimov, I. Farago and L. Vulkov (Eds), Proceedings of 5th International Conference, Numerical Analysis and its Applications 2012, Lozenetz, Bulgaria, June 2012, LNCS 8236, 2013, pp. 445-453.
7. N. Madden, M. Stynes: A Uniformly convergent numerical method for a coupled system of two singularly perturbed linear reaction-diffusion problems, IMA J. Numer. Anal. 23, 2003, pp.627-644.
8. S. Matthews: Parameter robust numerical methods for a system of two coupled singularly perturbed reaction-diffusion equations, Master Thesis, School of Mathematical Sciences, Dublin City University, 2000.
9. R. Mythili Priyadharshini, N. Ramanujam: Uniformly convergent numerical methods for a system of coupled singularly perturbed convection-diffusion equations with mixed type boundary conditions, Mathematical modelling and Analysis, 2013, iFirst:1-22.
10. P. Mahabub Basha, V. Shanthi: A uniformly convergent scheme for a system of two coupled singularly perturbed reaction-diffusion Robin type boundary value problems with discontinuous source term, submitted for publication.

* * *

Mahabub Basha Pathan/ Research Scholar (QIP)/ Department of Mathematics/
National Institute of Technology/ Tiruchirappalli/ Tamilnadu/ India/ pmbasha9@gmail.com
Shanthi Vembu/Assistant Professor/ Department of Mathematics/
National Institute of Technology/ Tiruchirappalli/ Taminadu,/India/vshanthi@nitt.edu