ON GENERALISED $b^{\#}$- CLOSED SETS

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Abstract: In the year 1996, Andrijivic [2] introduced and studied the concept of b - open sets. Ahmad Al omari et.al. [3] studied the concept of generalized b - closed sets. Recently Usha Parameswari et.al.[4] introduced the notion of $b^{\#}$- open sets and $b^{\#}$- closed sets. In this paper notion of a generalized $b^{\#}$ - closed set is introduced and its basic properties are discussed.

Keywords: b – closed, $b^{\#}$– closed, gb – closed.

Introduction: In the year 1996, Andrijivic [2] introduced and studied the concept of b - open sets. Ahmad Al omari et.al. [3] studied the concept of generalized b - closed sets. Recently Usha Parameswari et.al.[4] introduced the notion of $b^{\#}$- open sets and $b^{\#}$- closed sets. In this paper notion of a generalized $b^{\#}$ - closed set is introduced and its basic properties are discussed.

Preliminaries: Throughout this paper X denotes a topological space on which no separation axiom is assumed. For any subset A of X, *cl*(A) denotes the closure of A and *int* (A) denotes the interior of A in the topological space X.

Definition. 2.1. A subset A of space X is said to be
(i) b - open [2] if A ⊆ *cl*(*int* (A)) ∪ *int* (*cl*(A)).
(ii) b - closed [2] if *cl*(*int* (A)) ∩ *int* (*cl*(A)) ⊆ A.
(iii) $b^{\#}$ - open [4] if A = *cl*(*int* (A)) ∪ *int* (*cl*(A)).
(iv) $b^{\#}$- closed [4] if A = *cl*(*int* (A)) ∩ *int* (*cl*(A)).
(v) generalized closed [1] (briefly g – closed) if *cl* (A) ⊆ U whenever A ⊆ U and U is open.
(vi) generalized b - closed [3] (briefly gb - closed) if b*cl* (A) ⊆ U whenever A ⊆ U and U is open where b*cl* (A) is the intersection of all b – closed sets containing A.

Generalized $b^{\#}$ − Closed Sets: In this section generalized $b^{\#}$ - closed set is introduced and its properties are studied. The notation $b^{\#}$- closure of A denoted by $b^{\#}cl$(A) can be defined in the usual manner.

Definition 3.1. Let X be a space. A subset A of X is called generalized $b^{\#}$- closed (simply $gb^{\#}$- closed) if $b^{\#}cl$(A) ⊆ U whenever A ⊆ U and U is open.
The complement of a generalized $b^{\#}$- closed set is generalized $b^{\#}$- open (simply $gb^{\#}$ - open). The collection of all $gb^{\#}$ - closed (resp.$gb^{\#}$ - open) subsets of X is denoted by $Gb^{\#}C(X)$ (resp. $Gb^{\#}O(X)$).

Theorem 3.2. Every $b^{\#}$- closed set is $gb^{\#}$ - closed.

Proof. Let A be any $b^{\#}$- closed set in X such that A ⊆ U where U is open. Since A is $b^{\#}$- closed, $b^{\#}cl$(A) = A. Therefore $b^{\#}cl$(A) ⊆ U, that implies A is $gb^{\#}$- closed in X.
The converse of the above theorem is not true as seen from the following example.

Example 3.3. Let X = $\{a, b, c\}$ and let $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Then the family of all $b^{\#}$− closed subsets of X is $b^{\#}C(X) = \{\phi, X, \{a\}, \{b\}\}$ but the family of all $gb^{\#}$− closed sets of X is $Gb^{\#}C(X)=\{\phi, X, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$, then it is clear that $\{a, c\}$ is $gb^{\#}$− closed but not $b^{\#}$− closed.

Theorem 3.4. Every $gb^{\#}$- closed set is gb - closed.

Proof. Let A be a $gb^{\#}$- closed set in X such that A ⊆ U where U is open. Since A is $gb^{\#}$ closed, $b^{\#}$ *cl*(A) ⊆ U. Since b *cl*(A) ⊆ $b^{\#}$ *cl*(A), b *cl*(A) ⊆ U, that implies A is gb - closed in X.
The converse of the above theorem is not true as seen from the following example.

Example 3.5. Let X = $\{a, b, c\}$ and let $\tau = \{\phi, X, \{a\}\}$ then the family of all $gb^{\#}$− closed subsets of X is $Gb^{\#}C(X) = \{\phi, X\}$ but the family of all gb − closed sets of X is GbC(X) $=\{\phi, X, \{a, b\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$. Then it is clear that $\{b, c\}$ is gb − closed but not $gb^{\#}$− closed.
The following diagram summarizes the implications among the introduced concept and other related concepts.

$b^{\#}$ - closed ⟹ b – closed
⇓ ⇓
$gb^{\#}$ - closed ⟹ gb – closed

Theorem 3.6. If A is open and $gb^{\#}$- closed, then A is $b^{\#}$- closed and hence gb- closed.

Proof. Since A is open and $gb^{\#}$- closed, we have $b^{\#}cl$(A) ⊆ A. But A ⊆ $b^{\#}cl$(A). Therefore A= $b^{\#}cl$(A). A is $b^{\#}$- closed and gb- closed.

Theorem 3.7. Let A be a $gb^{\#}$ - closed subset of (X, τ). Then $b^{\#}cl$(A) - A does not contain any non - empty closed set.

Proof. Assume that A is $gb^{\#}$ - closed. Let F be a closed subset of $b^{\#}cl$(A) – A. A ⊆ X – F (or) F ⊆ $b^{\#}cl$(A). Since
X – F is open and since A is $gb^{\#}$- closed, we have, $b^{\neq}cl$(A) ⊆ X – F (or) F ⊆ X - $b^{\#}cl$(A) that implies F ⊆ $b^{\#}cl$(A) ∩ (X - $b^{\#}cl$(A)). Then F = ϕ. Thus F is empty.
The converse of the above theorem is not true as seen from the following example.

Example 3.8. Let X = $\{a, b, c\}$ and let $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and A = $\{b, c\}$. Then $b^{\#}cl$(A) − A = X − $\{b, c\}$ = $\{a\}$ does not contain non empty closed set but A is not a $gb^\#$ - closed set in X.

Corollary 3.9. If A is $gb^\#$ - closed set and open set of X, then A is $b^\#$- closed.

Proof. Since A is a $gb^\#$ - closed set and open, we have $b^\# cl(A) \subseteq U$. Also $A \subseteq b^\# cl(A)$ which implies $A = b^\# cl(A)$. A is $b^\#$- closed.

Theorem 3.10. For x ε X, the set X – {x} is $gb^\#$ - closed or open.

Proof. Suppose X – {x} is not open, then X is the only open set containing X – {x}. This implies $b^\# cl(X - \{x\}) \subseteq X$. Then X – {x} is $gb^\#$ - closed in X.

Example 3.11. The intersection of two $gb^\#$ -closed sets is generally not a $gb^\#$ - closed set.

Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{a\}\}$. If $A = \{a,b\}$ and B = {a,c} then A and B are each $gb^\#$ - closed but A ∩ B is not $gb^\#$ - closed.

Example 3.12. The union of two $gb^\#$ - closed sets is generally not a $gb^\#$ - closed set.

Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{a\},\{b\},\{a, b\}\}$. If $A = \{a\}$ and B = {b} then A and B are each $gb^\#$ - closed but A ∪ B is not $gb^\#$ - closed.

Corollary 3.13. Let A and B be $gb^\#$- closed sets in (X, τ) such that $cl(A) = b^\# cl(A)$ and $cl(B) = b^\# cl(B)$. Then A ∪ B is $gb^\#$- closed.

Proof. Let U be an open set such that $A \cup B \subseteq U$. Then since A and B be $gb^\#$- closed sets we have $b^\# cl(A) \subseteq U$ and $b^\# cl(B) \subseteq U$. Since $cl(A) = b^\# cl(A)$ and $cl(B) = b^\# cl(B)$ we have, $b^\# cl(A \cup B) \subseteq cl(A \cup B) = cl(A) \cup cl(B) = b cl(A) \cup b cl(B) \subseteq U$, which implies that A ∪ B is $gb^\#$- closed.

Definition 3.14. Let X be a topological space such that $B \subseteq A \subseteq X$. Then B is $gb^\#$ - closed relative to A if $b^\# cl_A(B) \subseteq U$ when $B \subseteq U$ and U is open in A where $b^\# cl_A(B) = b^\# cl(B) \cap A$.

Theorem 3.15. Let $B \subseteq A \subseteq X$ where A is $b^\#$ - closed and open set. Then B is $gb^\#$ - closed relative to A if and only if B is $gb^\#$ - closed in X.

Proof. Let $B \subseteq A \subseteq X$ where A is $b^\#$ - closed and open set. Suppose that B is $gb^\#$ - closed relative to X. Let $B \subseteq U$ where U is open in A. Then $U = G \cap A$ where G is open in X. Thus $B \subseteq G$. Since B is $gb^\#$ - closed relative to X, $b^\# cl(B) \subseteq G$. Then $b^\# cl(B) \cap A \subseteq G \cap A = U$. Thus $b^\# cl(B) \cap A \subseteq U$, that implies $b^\# cl_A(B) \subseteq U$. B is $gb^\#$ - closed relative to A.

Conversely suppose that B is $gb^\#$ - closed relative to A. Let $B \subseteq G$ where G is open in X. $B \subseteq G \cap A$. Since $G \cap A$ is open in A, $b^\# cl_A(B) \subseteq G \cap A$. Then $b^\# cl(B) \cap A \subseteq G \cap A$. Since A is $b^\#$ - closed, $b^\# cl(B) \subseteq b^\# cl(A) = A$ implies $b^\# cl(B) = b^\# cl(B) \cap A \subseteq G \cap A \subseteq G$. B is $gb^\#$ - closed relative to X.

Theorem 3.16. If A is $gb^\#$- closed and $\square \subseteq B \subseteq b^\# cl(A)$ then B is $gb^\#$- closed.

Proof. Let A be a $gb^\#$- closed set in X such that $\square \subseteq B \subseteq b^\# cl(A)$. Let U be a open set of X such that $B \subseteq U$. Then $A \subseteq U$. Since A is $gb^\#$- closed and $A \subseteq U$, we have $b^\# cl(A) \subseteq U$. Also $b^\# cl(A) = b^\# cl(B)$. Therefore $b^\# cl(B) \subseteq U$, that implies B is $gb^\#$- closed.

Example 3.17. In the real line R', every singleton set is $gb^\#$- closed.

Proof For let $x \in R'$. Fix an open set G such that $\{x\} \subseteq G$. Then $b^\# cl\{x\}$ = the intersection of $b^\#$ closed sets $= \cap_{\in > 0} (x - \in, x + \in) = \{x\} \subseteq G$. Thus every singleton set is $gb^\#$- closed.

Theorem 3.18. Every interval is $gb^\#$- closed in R.

Proof. Let A be an interval in R. If A is empty, nothing to prove. If A is an open interval of the form (-∞, a), (a, ∞), (a,b), a < b then by Example 5.11 of [4], A is $b^\#$- closed. Therefore by Theorem 3.2, A is $gb^\#$- closed.

Generalized $b^\#$- Open Sets: As seen in the previous section, the complement of a generalized $b^\#$- closed set is generalized $b^\#$- open. This leads to the definition of generalized $b^\#$- open set.

Definition 4.1. Let X be a space. A set A of X is called generalized $b^\#$- open (simply $gb^\#$- open) iff X – A is generalized $b^\#$- closed.

Theorem 4.2. A set A is $gb^\#$- open iff $F \subseteq b^\#\text{-} int(A)$ whenever F is closed and $F \subseteq A$.

Proof. Let A be a $gb^\#$- open set. Let F be a closed set such that $F \subseteq A$. Then $X - A \subseteq X - F$ where X – F is open. Since X – A is $gb^\#$- closed, $b^\#\text{-} cl(X - A) \subseteq X - F$ and hence $X - b^\#\text{-} int(A) \subseteq X - F$ that implies $F \subseteq b^\#\text{-} int(A)$. Conversely, assume that $F \subseteq b^\#\text{-} int(A)$ whenever $F \subseteq A$, F is closed. Suppose $X - A \subseteq U$ where U is open. Then $X - U \subseteq A$ where X – U is closed. By assumption, $X - U \subseteq b^\#\text{-} int(A)$ that implies $b^\#\text{-} cl(X - A) \subseteq U$. This proves that X – A is $gb^\#$- closed and hence A is $gb^\#$- open.

Theorem 4.3. Every $b^\#$- open set is $gb^\#$ - open.

Proof. Let A be any $b^\#$- open set in X such that $F \subseteq A$, where F is closed. Since A is $b^\#$- open, $A = b^\#\text{-} int(A)$. Therefore $F \subseteq b^\#\text{-} int(A)$ that implies A is $gb^\#$- open in X.

The converse of the above theorem is not true as seen from the following example.

Example 4.4. Let $X = \{a, b, c\}$ and let $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Then the family of all $b^\#-$ open subsets of X is $b^\# O(X) = \{\phi, X, \{a, c\}, \{b, c\}\}$ but the family of all $gb^\#-$ open sets of X is

$Gb^\# O(X) = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$

then it is clear that $\{a, b\}$ is $gb^\#-$ open but not $b^\#-$ open.

Theorem 4.5. Every $gb^\#$- open set is gb - open.

Proof. Let A be a $gb^\#$- open set in X such that $F \subseteq A$, where F is closed. Since A is $gb^\#$ - open, $F \subseteq b^\#$-

int (A). Since b *int*(A) ⊆ $b^\#$ *int*(A), F ⊆ $b^\#$- *int* (A), that implies A is gb – open in X.

The converse of the above theorem is not true as seen from the following example.

Example 4.6. Let X = $\{a, b, c\}$ and let $\tau = \{\phi, X, \{a\}\}$ then the family of all $gb^\#$– open subsets of X is $Gb^\#O(X)=\{\phi, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\}$ but the family of all gb – open sets of X is GbO(X) = $\{\phi, X, \{a\}, \{b\}, \{c\}, \{a,c\}, \{b,c\}\}$. Then it is clear that {c} is gb – open but not $gb^\#$– open.

The following diagram summarizes the implications among the introduced concept and other related concepts.

$b^\#$ - open ⟹ b – open
⇓ ⇓
$gb^\#$ - open ⟹ gb – open

Theorem 4.7. If F is closed and $gb^\#$ - open then F is $b^\#$ - open and hence gb – open.

Proof. Since F is closed and $gb^\#$ - open we have, F ⊆ $b^\#$- *int* (A). But $b^\#$- *int* (A) ⊆ F that implies F = $b^\#$- *int* (A). Therefore F is $b^\#$ - open and hence gb – open.

Theorem 4.8. If $b^\#$- *int* (A) ⊆ B ⊆ A and A is $gb^\#$- open, then B is $gb^\#$- open.

Proof. Let A be a $gb^\#$- open and $b^\#$- *int* (A) ⊆ B ⊆ A. Then X - A ⊆ X - B ⊆ X - $b^\#$- *int* (A) that implies X - A ⊆ X - B ⊆ $b^\#$- *cl* (X – A). Since X - A is $gb^\#$- closed, by Theorem. 3.16, X – B is $gb^\#$- closed. This proves that B is $gb^\#$- open.

Theorem 4.9. If A ⊆ X is $gb^\#$- closed, then $b^\#$- *cl* (A) – A is $gb^\#$- open.

Proof. Let A ⊆ X be $gb^\#$- closed and let F be a closed set such that F ⊆ $b^\#$- *cl* (A) – A. Then by Theorem.3.7, F = ϕ that implies F ⊆ $b^\#$- *int* ($b^\#$- *cl* (A) – A). This proves that $b^\#$- *cl* (A) – A is $gb^\#$- open.

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