A NUMERICAL METHOD FOR A SECOND ORDER SINGULARLY PERTURBED CONVECTION-DIFFUSION PROBLEM WITH DISCONTINUOUS SOURCE TERM

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Abstract: : In this paper a second order singularly perturbed convection–diffusion equation with a discontinuous source term is examined. Boundary and weak interior layers appear in the solution. A numerical method is constructed for this problem. Numerical results are presented which validate the theoretical results.

Keywords: Discontinuous source term,exponential fitting factor, finite difference scheme, singularly perturbed convection diffusion equation

Introduction: Singularly Perturbed Differential Equations (SPDE) appears in several branches of applied mathematics. Analytical and numerical treatment of these equations has drawn much attention of many researchers [1],[2]. In general, classical numerical methods fail to produce good approximations for these equations. Hence one has to go for non classical methods. A good number of articles have been appearing in the past three decades on non classical methods which cover mostly second order equations. Singularly perturbed second order problems are classified on the basis that how the order of the original differential equation is affected if one sets $\varepsilon = 0$ [1]. Here ε is a small positive parameter multiplying the highest derivative of the differential equation. We say that a Singular Perturbation Problem (SPP) is of convection- diffusion type if the order of the differential equation is reduced by 1, whereas it is called reaction-diffusion type if the order is reduced by 2. Various methods and applications are available in the literature in order to obtain numerical solution to Singularly Perturbed Differential equation with non smooth data [4] –[7]. In [8] the author derived Fitting Factor Finite Difference Method (FFFDM) to second order Singularly Perturbed Convection- Diffusion Problem (SPCDP) with continuous function f(x). Motivated by [8], [9], we follow an Exponential Fitting Factor Method (EFFM) to solve (SPCDP) in one dimension with a discontinuous source term. Throughout this paper, C denotes a generic constant that is independent of the singular perturbation parameter ε and N the dimension of the discrete problem.

2. Description of the problem.

A singularly perturbed convection-diffusion equation in one dimension with a discontinuous right hand side term is considered on the unit interval $\Omega = (0, 1)$. A single discontinuity in the right hand side term is assumed to occur at a point $d \in \Omega$. It is convenient to introduce the notation $\Omega^- = (0, d)$ and $\Omega^+ = (d, 1)$ and to denote the jump at d in any function with

$[u](d) = [u](d+) - [u(d-)$. The corresponding two point boundary value problem is

Find $u(x) \in C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ such that,

$$\left.\begin{array}{l} L_\varepsilon\big(u(x)\big) \equiv \varepsilon u''(x) + a(x)u'(x) = f(x) \\ u(0) = u_0 \ , u(1) = u_1, 0 < \varepsilon \ll 1, \\ a(x) > \alpha > 0, \text{for all } x \in (\Omega^- \cup \Omega^+) \\ |[f](d)| \le C, \end{array}\right\} (P_\varepsilon)$$

where $f \in C^2(\Omega^- \cup \Omega^+)$. If one assumes that a, f $\in C^1(\Omega)$ then a boundary layer appears in the solution near the point $x = 0$ and the solution $u(x) \in C^2(\Omega)$. In this paper, we construct and analyse a finite difference scheme with a suitable fitting factor for problems of the form (P_ε).

L_ε satisfies the following comparison principle on $\bar{\Omega}$.

We state here lemma 1 and 2 for the BVP (P_ε) which can be obtained by using the procedures adopted in [1],[2],[3].

Lemma1.(Comparison principle) Suppose that a function $u(x) \in C^0(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ satisfies

$u(0) \le 0$, $u(1) \le 0$, $L_\varepsilon u(x) \ge 0$,

$x \in (\Omega^- \cup \Omega^+)$ and $[u'(x)](d) \ge 0$ then

$u(x) \le 0$ for all $x \in \bar{\Omega}$.

Lemma2. Let u(x) be the soluiton of (P_ε), then

$$\|u(x)\|_{\bar{\Omega}} \le \max\{|u_0|, |u_1|\} \frac{1}{\alpha} \|f\|_{\bar{\Omega}\setminus d} \quad (1)$$

and $\|u^k(x)\|_{\bar{\Omega}\setminus d} \le C\varepsilon^{-k}, 1 \le k \le 3$. (2)

The bounds on the derivatives of the solution are obtained by decomposing the solution u(x) into regular (v(x)) and singular (w(x)) components as $u(x) = v(x) + w(x)$ consider $v(x) = v_0(x) + \varepsilon v_1(x) + \varepsilon^2 v_2(x)$.

where, $v_0, v_1, v_2 \in C^0(\Omega)$ and satisfy the following

$av_0'(x) = f, v_0(1) = u(1)$,

$av_1'(x) = -v_0''(x), \quad v_1(1) = 0$,

$x \in (\Omega^- \cup \Omega^+)$

$L_\varepsilon v_2(x) = -v_1''(x)$, $x \ne d$

$v_2(0) = v_2(d) = v_2(1) = 0$.

Choosing the smooth component

$v(x) \in C^0(\Omega)$ such that v(x) is the solution of

$$\left.\begin{array}{l} L_\varepsilon v(x) = f, \quad x \in (\Omega^- \cup \Omega^+) \\ v(0) = v_0(0) + \varepsilon v_1(0), v(1) = u(1) \\ v(d) = v_0(d) + \varepsilon v_1(d), \end{array}\right\} (3)$$

Lemma 3. (Farrell et al. [6]). For each integer k , satisfying $0 \le k \le 3$, the solution v(x) of (3) satisfies the following derivatives bounds.

$$\|v\| \le C, \ \|v^{(k)}\|_{\Omega^- \cup \Omega^+} \le C(1 + \varepsilon^{2-k}) \quad (4)$$

Also $|[v'](d)|, |[v''](d)| \le C$.

Define the layer component (w(x)) of the decomposition as follows:

$w(x) = w_1(x) + w_2(x)$ such that $w_1, w_2 \in C^0(\Omega)$,

where $w_1(x)$ is the boundary layer function satisfying

$L_\varepsilon w_1(x) = 0$, $x \in \Omega$, (5)

$w_1(0) = u(0) - v(0), \quad w_1(1) = 0$
and $w_2(x)$ is the interior layer function satisfying
$L_\varepsilon w_2(x) = 0, \quad x \in \Omega^- \cup \Omega^+$ (6)
$w_2(0) = 0, w_2(1) = 0,$
$[w_2'](d) = -[v'](d)$. ▪

Lemma 4. (Farrell et al [5]). For each integer k, satisfying $0 \le k \le 3$, the solution $w_1(x)$ of (5) satisfies the bounds $\left|w_1^{(k)}(x)\right| \le C\varepsilon^{-k}e^{-\alpha x/\varepsilon}, \quad x \in \Omega$, where C is a constant independent of ε.

Lemma 5. (Farrell et al. [5]). For each integer k, satisfying, $0 \le k \le 4$, the solution $w_2(x)$ of (6) satisfies the bounds

$$\left|w_2^{(k)}(x)\right| \le \begin{cases} C\left(\varepsilon^{1-k}e^{-\frac{\alpha x}{\varepsilon}}\right), & x \in \Omega^-, \\ C(\varepsilon^{1-k}e^{-\alpha(x-d)/\varepsilon}), & x \in \Omega^+, \end{cases}$$

where C is a constant independent of ε. ▪

In general the solutions has a boundary layer near $x = 0$ and a weak interior layer in the region to the right of $x = d$. Note that although $w_2 = O(\varepsilon)$ is small, it is of the same order throughout all of Ω^-. This contradicts with w_1 whose influence is only significant in the layer region near $x = 0$.

3. Discrete Problem: In this section we present an exponentially fitting finite difference scheme with a fitting factor $\boldsymbol{\sigma(\rho)}$ to obtain numerical solution for the BVP $(\mathbf{P_\varepsilon})$. As done in the case of continuous problem, the comparison principle and a uniform stability result are presented for the corresponding discrete problem.

Consider the SPBVP (P_ε) and its corresponding discrete problem defined in the following, in more general form
Find a mesh function $U(x_i)$ such that

$$\left.\begin{array}{l} L_\varepsilon^N U(x_i) \equiv \varepsilon\sigma(\rho)D^+D^-U(x_i) + \\ \quad a(x_i)D^+U(x_i) = f(x_i) \\ \quad \text{for all } x_i \in \Omega_\varepsilon^N \\ U(x_0) = u_0, U(x_N) = u_1 \\ D^+U(x_{N/2}) = D^-U(x_{N/2}) \end{array}\right\} (P_\varepsilon^N)$$

where,

$$D^+U(x_i) = \frac{U(x_{i+1}) - U(x_i)}{x_{i+1} - x_i},$$

$$D^-U(x_i) = \frac{U(x_i) - U(x_{i-1})}{x_i - x_{i-1}}$$

$$\Omega_\varepsilon^N = \left\{x_i : 1 \le i \le \frac{N}{2} - 1\right\} \cup \left\{x_i : \frac{N}{2} + 1 \le i \le N - 1\right\} \text{ and } x_{N/2} = d$$

$$\overline{\Omega}_\varepsilon^N = \{x_0, x_N, \Omega_\varepsilon^N\}$$

$$\sigma(\rho) = \frac{\rho a(0)}{e^{\rho(a(0))} - 1}, \quad \rho = \frac{h}{\varepsilon}. \quad (7)$$

Lemma 6. (Discrete Comparison Principle) Suppose that a mesh function $U(x_i)$ satisfies,
$U(x_0) \le 0, \; U(x_N) \le 0, \; L_\varepsilon^N U(x_i) \ge 0,$
for all $x_i \in \Omega_\varepsilon^N$ and $D^+U(x_{\frac{N}{2}}) - D^-U(x_{\frac{N}{2}}) \ge 0$
then $U(x_i) \le 0$ for all $x_i \in \overline{\Omega}_\varepsilon^N$.

4. Error analysis: Let us denote the error at each mesh point $x_i \in \overline{\Omega}_\varepsilon^N$ by $e(x_i) = (U(x_i) - u(x_i))$. To bound the nodal error $|e(x_i)|$, the solution $U(x_i)$ of the discrete problem (P_ε^N) is decomposed in a similar manner to the decomposition of the solution of the continuous problem (P_ε) as regular $V(x_i)$ and singular $W(x_i)$ components $U(x_i) = V(x_i) + W(x_i)$

Based on the proof derived in [1], [5] one can obtain the following error estimates.

$|V(x_i) - v(x_i)| \le Ch$ for all $x_i \in \Omega_\varepsilon^N$ (8)
$|W(x_i) - w(x_i)| \le Ch^{2/3}$ for all $x_i \in \Omega_\varepsilon^N$ (9)

From (8) and (9) it follows that
$|U(x_i) - u(x_i)| \le Ch^{2/3}$ for all $x_i \in \Omega_\varepsilon^N$ (10) where C is a constant independent of ε and N.

At the mesh point $x_{\frac{N}{2}} = d$,

$$\left|(D^+ - D^-)e(x_{N/2})\right| = \left|(D^+ - D^-)\left(U(x_{\frac{N}{2}}) - u(x_{\frac{N}{2}})\right)\right|$$

But $(D^+ - D^-)\,U(x_{\frac{N}{2}}) = 0$.

Therefore,

$$\left|(D^+ - D^-)e(x_{N/2})\right| = \left|(D^+ - D^-)\,u(x_{\frac{N}{2}})\right|$$
$$\le \left|D^+ - \frac{d}{dx}(u(d)\right| + \left|D^- - \frac{d}{dx}(u(d)\right|$$
$$\le \frac{Ch}{\varepsilon^2} \quad (11)$$

Theorem7. The solutions $u(x)$ of (P_ε) and $U(x_i)$ of (P_ε^N) satisfy the following bound
$|U(x_i) - u(x_i| \le Ch^{2/3}$ for $i = 1\,(1)\,N$.

Proof.
From (10) it follows that
$|U(x_i) - u(x_i)| \le Ch^{2/3}$ for all $x_i \in \Omega_\varepsilon^N$
Consider the discrete barrier function $\psi(x_i)$ [1] as

$$\psi(x_i) = Ch^{2/3} + \frac{Ch}{\varepsilon^2} \pm Ch^{2/3} + \frac{Ch}{\varepsilon^2}\varphi(x_i)$$

$$\text{where } \varphi(x_i) \begin{cases} \frac{{x_{i+1}}^2}{\varepsilon} e^{-\alpha\left(\frac{x_{i+1}}{\varepsilon}\right)}, & \forall\; 0 \le x_i \le \frac{N}{2} - 1 \\ \frac{{x_{i-1}}^2}{\varepsilon} e^{-\alpha\left(\frac{x_{i-1}}{\varepsilon}\right)}, & \forall\; \frac{N}{2} + 1 \le x_i \le 1 \end{cases}$$

Applying the discrete comparison principle to $\psi(x_i)$ we get $\left|U(x_i) - u(x_i)\right| \le Ch^{\frac{2}{3}}$ ▪

Note. [1] If variable fitting factor $\sigma_i(\rho) = \frac{\rho(a(x_i))}{e^{-\rho(a(x_i))} - 1}$ is considered instead of $\sigma(\rho)$ defined in (8) we obtain $\left|U(x_i) - u(x_i)\right| \le Ch$.

5. Numerical Results: In this section we verify experimentally the theoretical results obtained in the preceding section with an example.

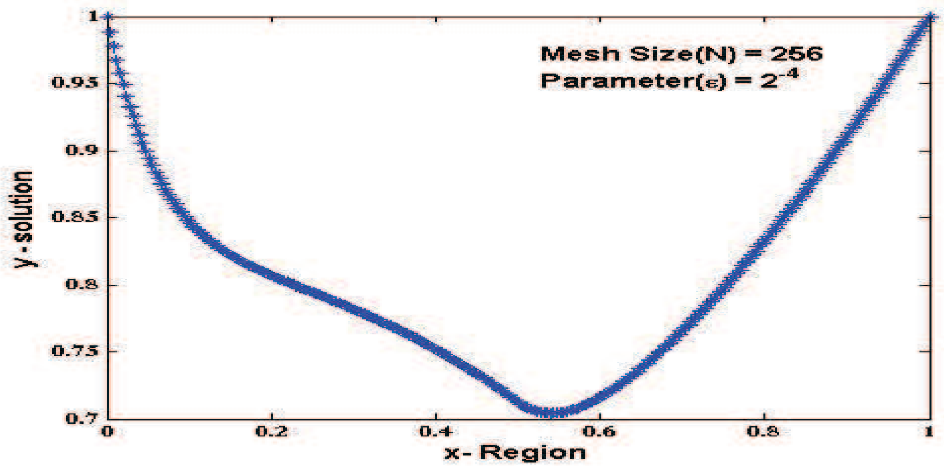

Fig. – Approximate solution for numerical Example.

N/ε	2^7	2^8	2^9	2^10	2^11	2^12	2^13
2^{-1}	3.8636E-04	1.9259E-04	9.6150E-05	4.8038E-05	2.4010E-05	1.2002E-05	6.0015E-06
2^{-2}	9.8405E-04	4.9017E-04	2.4462E-04	1.2219E-04	6.1065E-05	3.0525E-05	1.5258E-05
2^{-3}	1.5306E-03	7.5752E-04	3.7680E-04	1.8791E-04	9.3832E-05	4.6885E-05	2.3435E-05
2^{-4}	1.7879E-03	8.7503E-04	4.3276E-04	2.1519E-04	1.0729E-04	5.3572E-05	2.6768E-05
2^{-5}	1.9975E-03	9.5733E-04	4.6824E-04	2.3150E-04	1.1510E-04	5.7385E-05	2.8651E-05
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-20}	4.8370E-03	2.4300E-03	1.2178E-03	6.0964E-04	3.0500E-04	1.5254E-04	7.6283E-05
E_ε^N	4.8370E-03	2.4300E-03	1.2178E-03	6.0964E-04	3.0500E-04	1.5254E-04	7.6283E-05
ρ^N	9.9316E-01	9.9668E-01	9.9825E-01	9.9915E-01	9.9962E-01	9.9975E-01	

Example. Consider the BVP of (P_ε) with,

$$a(x) = 1,\ \ f(x) = \begin{cases} -x & \text{for} \quad x \le 0.5 \\ x & \text{for} \quad x > 0.5 \end{cases}$$

and $u(0) = u(1) = 1$

The errors and order of convergence are estimated using the double mesh principle [4]. Define the double mesh differences to be

$E^N = |U(x_i)^N - U(x_i)^{2N}|$ and

$E_\varepsilon^N = \max(E^N)$

From these quantities the order of convergence is computed by $\rho^N = \log_2 \frac{E_\varepsilon^N}{E_\varepsilon^{2N}}$

Table - Computed maximum point wise error E_ε^N and order of convergence ρ^N for the example.

Conclusion: We have presented a numerical method to solve a second order singularly perturbed convection diffusion problem with discontinuous source term. Due to discontinuity a weak interior layer occurs. Error analysis is proved and we have obtained almost first order convergence.

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