ON FRECHET SPACES OF DISTRIBUTIONS

M.P.SINGH

Abstract: In this paper , to obtain some results in Fourier Analysis for non-normable spaces, Frechet spaces of distributions are defined. Such spaces are named as FD-Spaces. It is shown that a homogeneous FD-Space and its strong*dual admit Convolution with measures. A relationship between convolution and translations is also established for these spaces .

Keywords: Convolution, fourier Series, homogeneous, strong* dual, translation invariant.

Introduction: In [7],[8] and [9] some results of Fourier Analysis known for $L^p\left(1\le p\le\infty\right)$, C and Orlicz spaces have been generalized to Banach Spaces of distributions and Convolutable Banach spaces of distributions . But these results can not be applied directly to some important non-normable spaces .To over come this deficiency in this paper, we define Frechet spaces of distributions (FD-Spaces).If E is a homogeneous FD-Space and E* its strong*dual , for $f\in E$, $F\in E^*$ and $\mu\in M$ (The space of measures), it is shown that $\mu * f\in E$, $\mu * F\in E^*$ and $f*F\in C$. It is also proved that $\mu * f$ is the limit in E of finite linear combinations of translates of f.

2. Definitions and Notations: We refer to [2],[5],[6] and [11] for all standard definitions , notations and assumptions . all our functions and distributions are assumed to be defined on the circle group G . By M we denote the space of all complex regular Borel measures on G.

2.1 Definition. Let D(with strong* topology) denote the space of all distribution on G. We call a Frechet space E, an FD-space , If it can be continuously embedded into **D,** and regarded as a sub space of D, it satisfies the following properties:

(2.1.1) $C^\infty\subset E$

(2.1.2) $f\in E\Rightarrow T_x f\in E \quad\forall\quad x\in G$ and $\{T_x\}_{x\in R}$ is an equicontinous family of operators on E .

(2.1.3) $f\in E\Rightarrow \overset{\vee}{f}\in E$, where $\overset{\vee}{f}\left(u\right)=f\left(\overset{\vee}{u}\right)$ $\forall\ u\in C^\infty$ and $\overset{\vee}{u}\left(t\right)=u\left(-t\right)$ for all t in G .

From now onward E will denote an FD-space and E* will denote its strong* dual . Strong * topology of E* is denote by $\beta\left(E^*,E\right)$ [11,chapter 10]. All Banach Spaces like $L^p(1\le p\le\infty)$, C ,M etc are FD-Spaces , where as C^∞ is a non normable FD-space . To claim our extension fruitful we gives an example of an FD-space which is different from C^∞.

3. A non normable FD- space which is different from C^∞: Let $f\in D$ and $n\in N$ (the set of natural numbers), define

$$p_n\left(f\right)=\sum_{i\in Z}^{n} a_n\left(i\right)\left|\hat{f}\left(i\right)\right| \text{ , Where ,}$$

$$a_n\left(i\right)=\left(1+\left|i\right|\right)^{\frac{1}{2}+\frac{1}{2^2}+\dots\dots+\frac{1}{2^n}}$$

let $E=\left\{f\in D/\,p_n\left(f\right)<\infty\forall n\in N\right\}$

We see that $\{p_n\}_{n=1}^\infty$ is a countable family of seminorms on E which defines on E a locally convex metrizable topology , the metric is induced by

$$\left\|f\right\|_E=\sum_{k=1}^\infty\frac{2^{-k}p_k\left(f\right)}{1+p_k\left(f\right)} \qquad (3.1)$$

To show that E is a Frechet space, let $\{f_m\}_{m=1}^\infty$ be a Cauchy sequence .

Then , for $n\in N$ and $\varepsilon>0$, $\exists\ l>0$ such that $p_n\left(f_m-f_k\right)<\varepsilon \quad\forall\quad m,k\ge l$. That is

$$\sum_{i\in Z}a_n\left(i\right)\left|\hat{f}_m\left(i\right)-\hat{f}_k\left(i\right)\right|<\varepsilon\quad\forall m,k\ge l$$

$$\Rightarrow\quad\left|\hat{f}_m\left(i\right)-\hat{f}_k\left(i\right)\right|<\varepsilon\quad\forall\ m,k\ge l$$

Hence , $\left\{\hat{f}_m(i)\right\}_{m=1}^\infty$ is a Cauchy sequence in C (The field of complex numbers).

Therefore, for each $i\in Z$, there exists c_i in C such that

$$\lim_{m\to\infty}f_m\left(i\right)=c_i$$

Now fix $k=n_0>l$, Then

$$\sum_{|i|\le l'}a_n\left(i\right)\left|\hat{f}_m\left(i\right)-\hat{f}_{n_0}\left(i\right)\right|<\varepsilon$$

holds for all positive integer l' and for all $m \geq l$. Taking the limit as $m \to \infty$,
we get .

$$\sum_{|i| \leq l'} a_n(i)\left|c_i - \hat{f}_{n_0}(i)\right| < \varepsilon$$ for all positive integers l' .Hence,

$$\sum_{i \in Z} a_n(i)\left|c_i - f_{n_0}\right| < \varepsilon \quad , \quad (3.2)$$

keeping in mind that $f_{n_0} \in E$ we can deduce from (3.2) that for all $n \in N$

$$\sum_{i \in Z} a_n(i)\left|c_i\right| < \infty \quad , \quad (3.3)$$

Now consider $g(x) = \sum_{k \in Z} c_k e^{ikx}$, by (3.3) $\sum_{k \in Z}\left|c_k\right| < \infty$ and hence $g \in C$ and $\hat{g}(k) = c_k$.

Now again by (3.3), $g \in E$, by (3.2) $f_m \to g$ in E as $m \to \infty$. Hence E is a Frechet space . E and D(with strong* topology) are Frechet and webbed spaces respectively [4;14.5.3,p.322] , By the application of closed graph theorem [4;14.7.1,p 324] it follows that inclusion map of E into D is continuous.

By definition of E, $C^\infty \subset E$. Also if $f \in E$,than $T_a f \in E$ for each $a \in G$ since $(T_a f) = p_n(f) \quad \forall \quad n \in N$. $\{T_a\}_{a \in R}$ is an equicontinous family of operators on E. Further , $f \in E \Rightarrow \check{f} \in E$ as $p_n(f) = p_n(\check{f})$. This proves that E is an FD- space .

Next we show that E is not normable . If possible let E be normable , then E has a balanced , convex and bounded neighborhood B of 0 . Now, for some $n \in N$ and $\varepsilon > 0$,

$$\left\{f \in E \,/\, p_n(f) < \varepsilon\right\} \subset B$$

Since B is bounded , there exists M such that $p_{n+1}(f) \leq M \quad \forall \quad f \in B$
Hence

$$p_n(f) < \varepsilon \Rightarrow p_{n+1}(f) \leq M. \quad (3.4)$$

Find k > 0 such that $(1+k) > \left(\dfrac{2M}{\varepsilon}\right)^{n+1}$; and take

$$f = \frac{1}{a_n(k)} \frac{\varepsilon}{2} . e_k .$$

Then $p_n(f) < \varepsilon$ but $p_{n+1}(f) > M$,which contradicts (3.4). Hence E is not a normed space .

4. Some Preliminary Results:

4.1 Lemma. Let E be an FD-space, Then

(i) The inclusion map i: $C^\infty \to E$ is continuous .

(ii) $f \to \check{f}$ is continuous .

(iii) If , for $u \in C^\infty$, we define

$$ju(f) = f(u) \text{ for all } f \in E,$$

Then $j : C^\infty \to E^*$ (with strong* topology) is one to one and continuous .

Proof: (i) Let $u_n \to u$ in C^∞ and $i(u_n) = u_n \to v$ in E as $n \to \infty$. Since E is continuously embedded into D , $u_n \to v$ in D as $n \to \infty$.Hence $\hat{u}_n(k) = \hat{v}(k)$ for all k in Z. But $\hat{u}_n(k)$ also tends to $\hat{u}(k)$ for all k in Z therefore $\hat{u}_n(k) = \hat{v}(k)$ for all k in Z, and hence u=v . Closed graph theorem yields that i is continuous.

(ii) This can also be proved as (i) above.

(iii) To justify our definition , let us first show that $ju \in E^*$. Linearity of ju is obvious . Since E is continuously embedded into D, $f_n \to f$ in E implies that $f_n \to f$ in D as $n \to \infty$. Therefore $f_n(u) \to f(u)$ as $n \to \infty$ for every $u \in C^\infty$.Thus, $ju(f_n) \to ju(f)$ as $n \to \infty$. Hence ju is a continuous linear functional on E. For continuity of j, let $u_n \to u$ in C^∞ and $ju_n \to g$ in E* as $n \to \infty$.Then for every f in E,

$$g(f) = \lim_{n \to \infty} ju_n(f) = \lim_{n \to \infty} jf(u_n) = u(f) = ju(f)$$

$$\forall \; f \in E .$$

Hence g= ju . Since C^∞ and E* (with strong* topology) are Frechet and webbed spaces respectively, by closed graph theorem [4;14.5.3, p 322] , j is continuous .

Now suppose $ju_1(f) = ju_2(f)$ for u_1, u_2 in C^∞ .Then

$ju_1(f) = ju_2(f)$ for all $f \in E$

$u_1(f) = u_2(f)$ for all $f \in E$

taking $f = e_{-k}$, we obtained $\hat{u}_1(k) = \hat{u}_2(k)$ for all k in Z.

Hence $u_1 = u_2$, therefore j is one one .

4.2 Remark: In view of part (iii) of above Lemma, given any $u \in C^{\infty}$, we identify ju with u, so we can write f(u)=u(f) for f in E.

4.3 Lemma: Let E be an FD-Space .Then

(i) P (the set of all trigonometric polynomials) is dense in E if and only if C^{∞} is dense in E.

(ii) Let C^{∞} be dense in E. For $F \in E^*$ defines $i'F(u) = F(u)$, for each $u \in C^{\infty}$,

then i' is one to one continuous linear transformation from E* into D. Thus in this case, E* is continuously embedded into D ,and can be regarded as a subspace of D.

Proof: Necessity is obvious as $p \subset C^{\infty}$, sufficiency is proved by using Lemma 4.1 and the fact that p is dense in C^{∞}. Let d be the metric on E induced by

$$\| f \|_E = \sum_{k=1}^{\infty} \frac{2^{-k} \, p_k(f)}{1 + p_k(f)} \quad (4.1)$$

$\{p_k\}_{k=1}^{\infty}$ is countable family of seminorms on E, which defines the locally convex topology of E. Given $f \in E$ and $\varepsilon > 0$, $\exists$ u in C^{∞} such that

$$d(f,u) < \frac{\varepsilon}{2}$$

Also for u in C^{∞} we have

$$\|\sigma_n u - u\|_{\infty} \to 0 \text{ as } n \to \infty$$

Therefore $\sigma_n u \to u$ in E is as $n \to \infty$. Since inclusion of C^{∞} into E is continuous.

So ,for $\varepsilon > 0$, there exists $l > 0$ such that $d(\sigma_n u, u) < \frac{\varepsilon}{2}$ for all $n \geq l$

Hence $d(\sigma_n u, u) \leq d(f,u) + d(\sigma_n u, u) < \varepsilon$ for all $n \geq l$,

therefore p is dense in E.

(ii) First we note that (E,E*) and (C^{∞},D) are dual pairs , and i' is the operator dual to $i : C^{\infty} \to E$.

By [11; p 164], i' is $\sigma(E^*,E) - \sigma(D,C^{\infty})$ continuous , by [11; p 169] , it is $\beta(E^*,E) - \beta(D,C^{\infty})$ continuous and then by [11; p 167], i' is one to one .

Thus E* with strong* topology is continuously embedded into D.

Homogeneous Banach subspace B of L^1, defined on the circle group G, having a norm $\|\bullet\|_B \geq \|\bullet\|_1$ are discussed in [3] and many results of Fourier analysis in these spaces are generalized to homogeneous BD-spaces in [7]and [8] . To obtain these results for non normable spaces, we further generalize them to homogenous FD-Spaces.

5. Homogeneous FD-Spaces:

5.1. Definition. Let E be a translation invariant locally convex space and let T_x denote the translation operator for each x in R. Then E is said to be homogeneous if $x \to x_0$ in R implies $T_x f \to T_{x_0} f$ for each f in E.

5.2. Lemma. Let E be an FD-space and $\{k_n\}_{n=1}^{\infty}$ a summability kernel. Let ϕ be an E-valued continuous function on G. Then

$$\lim_{n\to\infty} \frac{1}{2\pi} \int k_n(t) \varphi(t)\, dt = \phi(0).$$

Proof: Since k_n and ϕ are continuous on G, the vector valued integral $\int k_n(t)\varphi(t)\,dt$ is defined and lies in E [6; Theorems 3.25 & 3.27]. For $0 < \delta < \pi$, we have

$$\frac{1}{2\pi}\int k_n(t)\big(\phi(t)-\phi(0)\big)\,dt = \frac{1}{2\pi}\int_{-\delta}^{\delta} k_n(t)\big(\phi(t)-\phi(0)\big)\,dt + \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} k_n(t)\big(\phi(t)-\phi(0)\big)\,dt$$

Let p be a continuous seminorms on E. Then , by.[6 ; theorem 3.3],

$$p[\frac{1}{2\pi}\int_{-\delta}^{\delta} k_n(t)\big(\phi(t)-\phi(0)\big)\,dt] \leq \frac{1}{2\pi}\int_{-\delta}^{\delta} |k_n(t)|\, p\big[\phi(t)-\phi(0)\big]\,dt$$

Since ϕ and p are continuous and $\int |k_n(t)|\,dt \leq cons\tan t$, given $\varepsilon > 0$, there exists a $\delta > 0$ (depending upon p and ϕ) such that $\frac{1}{2\pi}\int_{-\delta}^{\delta} |k_n(t)|\, p\big[\phi(t)-\phi(0)\big]\,dt < \frac{\varepsilon}{2}$ for all n. Also,

$$p[\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} k_n(t)\big(\phi(t)-\phi(0)\big)\,dt] \leq \max_{t\in[0,2\pi]} p\big[\phi(t)-\phi(0)\big] \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} |k_n(t)|\,dt$$

.

The above inequality together with the fact $\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} |k_n(t)|\,dt \to 0$ as $n \to \infty$, yields that $p[\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} k_n(t)\big(\phi(t)-\phi(0)\big)\,dt]$ can be made less than $\frac{\varepsilon}{2}$ for sufficiently large value of n. Combining this with the inequality obtained above , we see that

$$\lim_{n\to\infty} p[\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} k_n(t)\big(\phi(t)-\phi(0)\big)\,dt] = 0.$$

The above equality holds for each continuous seminorms p on E. Hence,

$\lim_{n\to\infty}\frac{1}{2\pi}\int k_n(t)\phi(t)\,dt=\phi(0)$ in E.

5.3. Theorem: Suppose E is a homogeneous FD-Space .Then $\lim_{n\to\infty}\sigma_n f=f$ in E, for every f in E, where $\sigma_n f$ denotes the n-th cesaro sum of the Fourier series of f .

Proof: Since E is homogenous, $t\to T_t f$ is an E-valued continuous function on G. By [6; theorems 3.25 & 3.27], the vector valued integral $\int e_k(t)T_t f dt$ is defined in E for each k.

Now for every n,

$$(f*e_k)^\wedge(n)=\hat{f}(n)\hat{e}_k(n)$$

$$= f(e_{-n})\frac{1}{2\pi}\int e_k(t)e^{-int}dt$$

$$=\frac{1}{2\pi}\int e_k(t)T_t f(e_{-n})dt$$

$$=[\frac{1}{2\pi}\int e_k(t)T_t f dt](e_{-n})$$

$$=\left(\frac{1}{2\pi}\int T_t f e_k(t)\,dt\right)^\wedge(n).$$

Hence ,

$$f*e_k=\frac{1}{2\pi}\int T_t f e_k(t)\,dt \quad \forall \quad k\in Z. \qquad (5.1)$$

Therefore ,

$$\sigma_n f=f*k_n=\frac{1}{2\pi}\int k_n(t)T_t f dt,$$

Where k_n is the n-th Fejer's Kernel .

Now , by the above Lemma, $\sigma_n f\to f$ in E as $n\to\infty$.

5.4. Theorem: An FD-Space E is homogeneous if and only if C^∞ is dense in E .

Proof: One part is clear from Theorem 5.3 as $\sigma_n f\in C^\infty$ for every n. Conversely, suppose C^∞ is dense in E . For any g in C^∞, $T_a g\to T_{a_0}g$ in C^∞ as $a\to a_0$ in R, and therefore , by Lemma 4.1 (i) , $T_a g\to T_{a_0}g$ in E as $a\to a_0$ in R. That is,

$$d\left(T_a g,T_{a_0}g\right)\to 0 \text{ as } a\to a_0 \text{ in R} \qquad (5.2)$$

for every g in C^∞ , where d is the metric induced by (4.1).

Let $f\in E$ and $\varepsilon>0$. Since $\{T_a\}_{a\in R}$ is an equicontinous family of operators on E by [6; Theorem 2.4] , $\exists$ $\delta>0$ such that

$$d(f,g)<\delta\Rightarrow d\left(T_a f,T_a g\right)<\frac{\varepsilon}{3} \quad \forall \quad a\in R.$$

(5.3)

Since C^∞ is dense in E , we may fix a function g in C^∞ such that $d(f,g)<\delta$.

Now ,

$$d\left(T_a f,T_{a_0}f\right)\le d\left(T_a f,T_a g\right)+d\left(T_a g,T_{a_0}g\right)+d\left(T_{a_0}f,T_{a_0}g\right)$$

Hence , by (5.2) and (5.3) ,

$$d\left(T_a f,T_{a_0}f\right)<\varepsilon$$

When ever a is close enough to a_0 . Hence , it follows that $T_a f\to T_{a_0}f$ in E as $a\to a_0$, which shows that E is homogeneous .

5.5 Theorem: Let E be a homogenous FD-Space . Then, for $\mu\in M$, $f\in E$ and $F\in E^*$,

(i) $F*f\in C$;

(ii) $\mu*F\in E^*$, where $\mu*F$ is defined by

$$\mu*F(u)=\mu\left[\left(F*\check{u}\right)^\vee\right] \quad \text{for all } u\in E;$$

(iii) $\mu*f\in E$.

Proof: (i) By Lemma 4.1 (iii) , E* can be regarded as a subspace of D . Therefore $F*f$ is defined as a distribution. Let us define $g(x)=F\left(T_x\check{f}\right)$ for all real x. Since E is homogeneous , we can find a sequence $\{u_n\}_{n=1}^\infty$ consisting of trigonometric polynomials such that for any continuous seminorms p on E , p(u_n-f) $\to 0$ as $n\to\infty$. Because of the continuity of F , the equicontinuty of $\{T_x\}_{x\in R}$ and the continuity of operation of crescent on E , we can find the continuous seminorm p on E [12; Theorem 1,p 42] such that

$$\left|F*u_n(x)-g(x)\right|=\left|F\left[T_x\left(\check{u}_n-\check{f}\right)\right]\right|\le p\left(u_n-f\right)$$

This shows that $\left|F*u_n(x)-g(x)\right|\to 0$ (uniformly in x) as $n\to\infty$.

But $F*u_n\in C$ for all n . Hence $g\in C$. Further , since E is continuously embedded into D as $n\to\infty$. By [2;II,12.6.6] $F*u_n\to F*f$ in D as $n\to\infty$, hence $g=F*f$. This proves part (i) of the theorem .

(ii) $\mu*F$, defined by

$$(\mu * F)(u) = \mu\left[\left(F * \overset{\vee}{u}\right)^{\vee}\right]$$

For all $u \in E$, is clearly a linear functional on E. By part (i) above and the closed graph theorem, it is easy to see that $u \to \left(F * \overset{\vee}{u}\right)^{\vee}$ is continuous from E to C. Also, μ is a continuous linear functional on C as M is the dual space of C. So $\mu * F$, being the composition of two continuous functions, is also continuous. Hence $\mu * F \in E^*$.

(iii)Since E is homogeneous, $\phi(x) = T_x f$, $x \in G$ is continuous on G and the vector valued integral $\left(\int T_x f d\mu'(x)\right)$ makes sense in E [6; Theorems 3.25 & 3.27], where μ' is a regular complex Borel measure Which represents μ as envisaged by the Riesz-representation theorem [5; theorem ;6.19]. To complete the proof, we shall show that

$$\mu * f = \frac{1}{2\pi}\left(\int_G T_x f d\mu'(x)\right)$$

By Lemma 4.1 (iii), $e_{-k} \in E^*$ for all $k \in Z$. Therefore

$$\left[\frac{1}{2\pi}\int_G T_x f d\mu'(x)\right]^{\wedge}(k) = \left[\frac{1}{2\pi}\int_G T_x f d\mu'(x)\right](e_{-k})$$

$$= \frac{1}{2\pi}\int_G (T_x f)(e_{-k}) d\mu'(x)$$

$$= \hat{f}(k)\frac{1}{2\pi}\int e^{-ikx} d\mu'(x)$$

$$= \hat{f}(k)\hat{\mu}(k) = (\mu * f)^{\wedge}(k)$$

This shows that $\mu * f = \frac{1}{2\pi}\int T_x f d\mu'(x)$. Hence $\mu * f \in E$ for $f \in E$ and $\mu \in M$.

5.6. Theorem: An FD-Space E is homogeneous if and only if, the series $\sum_{-\infty}^{\infty} \hat{F}(n)\hat{f}(-n)$ is (C,1) –sum able to F(f) for every f in E and F in E*.

Proof: First suppose that E is homogeneous $f \in E$ and $F \in E^*$. By Theorem 5.3 $\sigma_n f \to f$ in E as $n \to \infty$. Hence $F(\sigma_n f) \to F(f)$ as $n \to \infty$ where $F(\sigma_n f)$ is nothing but n-th Cesaro –sum of the series $\sum_{-\infty}^{\infty} \hat{F}(n)\hat{f}(-n)$.

Conversely, suppose that for every f in E and F in E*, $\lim_{n\to\infty} F(\sigma_n f) = F(f)$.This shows that trigonometric polynomials are weakly dense in E and hence strongly dense in E [6;theorem 3.12]. By Theorem 5.4, E is homogeneous.

5.7. Theorem: Let E be a homogeneous FD-Space. Then any distribution F is in E* if and only if, the series $\sum_{-\infty}^{\infty} \hat{F}(n)\hat{f}(-n)$ is (C,1)-summable for every f in E.

Proof: Direct assertion is proved in the previous theorem.

Conversely, let $\lim_{n\to\infty} F(\sigma_n f) = \lim_{n\to\infty} \sigma_n F(f)$ exist for each f in E. Define H on E by

$$H(f) = \lim_{n\to\infty} F(\sigma_n f) \quad \forall \quad f \in E. \qquad (5.4)$$

H is clearly linear and by [6;Theorem 2.8,p.45], it is continuous. Next, by taking $f = e_{-k}$ we obtain that $\hat{F}(k) = \hat{H}(k)$ for all k in Z. Hence $H = F \in E^*$.

5.8 Theorem: Let E be weakly sequentially complete homogeneous FD- space. Then any distribution f is in E, if and only if, the series $\sum_{-\infty}^{\infty} \hat{F}(n)\ \hat{f}(-n)$ is (C,1) summable for each F in E*.

Proof: One part is already proved in Theorem 5.6. To show the other part, let us suppose the series $\sum_{-\infty}^{\infty} \hat{F}(n)\ \hat{f}(-n)$ is (C-1) summable, i.e., $\lim_{n\to\infty} F(\sigma_n f)$ exists for every F in E*. Since $\sigma_n f \in E$ for each n, and E is weakly sequentially complete, there exists h in E such that

$$\lim_{n\to\infty} F(\sigma_n f) = F(h) \quad \forall \quad F \in E^*.$$

Now, replacing F by e_{-k} in the above equality, we obtain that $\hat{f}(k) = \hat{h}(k)$ for each k in Z. Hence f=h, and therefore $f \in E$.

6. Convolutions and Translations: A relationship between Convolutions and Translation operators is established in [7] for homogeneous Banach spaces of distributions. In this section we extend that result to homogeneous FD- spaces. But first we prove a slightly modified version of [2:II,11.2.1] .

6.1 Let E be an FD-Space, for any $f \in E$ let Z_f denotes the set of all those integers n for which $\hat{f}$ (n) =0 . For any subset S of E , we write

$Z_S = \bigcap \{ Z_f \mid f \in S \}$

6.2. Theorem: Let E be a homogeneous FD- space and I a translation invariant ($T_a f \in I$ for every $f \in I$ and each $a \in G$) closed linear subspace of E. Then $Z_I \subset Z_f$ if and only if $f \in I$.

Proof: Converse part is evident . For the direct assertion , let $f \in E \setminus I$. then , by [6:Theorem 3.5,p.59] , there exists $F \in E^*$ such that

F(f) =1 and F(g)=0 $\forall \; g \in I$. (6.1) Since I is translation invariant ,

$$F(T_a g) = 0 \quad \forall \; g \in I \text{ and each } a \in G . \quad (6.2)$$

Further, $F * \check{g} \in C$ for every $g \in E$ [see Theorem 5.3 (i)]. So, define $h_g(x) = F * \check{g}(x) = F(T_x g)$ for each $x \in G$. Clearly h_g is continuous and bounded. Moreover, using [6; Theorems 3.25 & 3.27] and(5.1) ,

$$\hat{h}_g(n) = F\left(\frac{1}{2\pi}\int (T_x g) e^{-inx} dx\right) = F(g * e_{-n}) .$$

Thus,

$$\hat{h}_g(n) = \hat{g}(-n) F(e_{-n}) . \quad (6.3)$$

Now in view of (6.2), (6.3) entails that $F(e_n) = 0$ whenever $g \in I$ and $\hat{g}(n) \neq 0$. Therefore, $F(e_n) = 0$ for all $n \in Z \setminus Z_I$. On the other hand Theorem 5.6 gives that

$$F(f) = \lim_{N \to \infty} \sum_{|n| \le N} \left(1 - \frac{|n|}{N+1}\right) \hat{f}(n) F(e_n) . \quad (6.4)$$

Since $Z_f \supset Z_I$ and $F(e_n) = 0 \;\; \forall \;\; n \in Z \setminus Z_I$, it follows from (6.4) that $F(f) = 0$. This contradicts the fact , established in (6.1) , that $F(f) = 1$. Hence $f \in I$.

6.3 Theorem: Let E be a homogeneous FD – space . Let $f \in E$ and $\mu \in M$, then $\mu * f$ is the limit in E of finite linear combination of translates of f .

Proof: Let us denote by $\overline{V_f}$ the closed linear subspace of E generated by $\{T_a f\}_{a \in R}$. That is, $\overline{V_f}$ is the closure in E of the set of all finite linear combination of translates of f . It is obviously a translation invariant closed linear subspace of E . Let $n \in Z_{\overline{V_f}}$, then $\hat{f}(n) = 0$ and $(\mu * f)^\wedge(n) = \hat{\mu}(n)\hat{f}(n) = 0$. This shows that $n \in Z_{\mu * f}$. Hence $Z_{\overline{V_f}} \subset Z_{\mu * f}$. By the application of Theorem 6.2, $\mu * f \in \overline{V_f}$. Hence $\mu * f$ is the limit in E of finite linear combination of translates of f .

References

1. N .Dunford and J.T. Schwartz , " Linear operators " , part I, inter science publishers , Inc , New York, 1958.
2. R.E. Edwards, "Fourier series" Vols. I ,II , Springer verlag , New York , 1979 , 1982.
3. Y. Ketznelson , "An introduction to Harmonic analysis" , John Wiley and Son's Inc., New York, 1968.
4. L. Narici and Edward Beck enstein , " Topological Vector Spaces" , Marcel Dekker , Inc. , New York , 1985.
5. W . Rudin , " Real and Complex Analysis " , Mc Graw Hills, Inc. New York , 1966.
6. W . Rudin " Functional Analysis " , Mc Graw Hills , Inc. , New York , 1973.
7. R. Sinha and Vishnu Kant , Homogeneous Banach Spaces of distributions, Comment . Math. Prace Mat. , 23 (1983) , No. 2, 309-314.
8. R.P. Sinha and Vishnu Kant , on Banach spaces of distribution , Bull. Soc. Math. Belg.Ser.B,41(1989) No. 3, 295-305.
9. R.P. Sinha and J.K. Nath , on Convolutable Banach spaces of distribution , Bull. Soc.Math. Belg. Ser. B , 41(1989) No. 2, 229-237.
10. H.C. Wang, "Homogeneous Banach algebras" Marcel Dekker INC, New York, lecture notes in pure and app. Math, vol29, 1977.
11. A.Wilansky,"Modern Methods In Topological Vector Spaces " , Mc Graw Hills , Inc, New York ,1978.
12. K. Yosida , " Functional Analysis " , Springer-Verlag , New York , 1980.

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Deptt. of Mathematics / M.L.B. Govt. College of Excellence Gwalior(M.P.) /India/mps_mlbg@rediffmail.com