STABILITY AND BIFURCATION ANALYSIS OF A HARVESTED PREDATOR-PREY MODEL

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Abstract: In the present paper, we discuss the stability and bifurcation of a predator-prey model with nonlinear harvesting in predator and zero natural death rate of predator. The stability of the equilibrium points has been discussed by linearization technique. We have shown that the proposed system undergoes saddle-node bifurcation. We have also found the bionomic equilibrium point. Numerical simulation has been given to support the analytical results which have been found here.

Keywords: bionomic equilibria, harvesting, MSY, saddle-node bifurcation.

Introduction: The first order nonlinear ordinary differential equations play a very important role in the study of interactions between predators and their prey. The Lotka [4] and Volterra [7] were the first who gave the first mathematical model to study these interactions, known as Lotka-Voletrra predator-prey model. This model consists of a system of two nonlinear first order differential equations. The exploitation of biological resources and the harvesting of populations are commonly practiced in fishery, forestry, and wildlife management. The effect of harvesting on the dynamics of predator-prey systems and the role of harvesting in the management of renewable resources have attracted great attention, Clark [2] studied the first harvested predator-prey model. Brauer and Soudack [1] studied the global behavior of a predator-prey model under constant rate predator harvesting. Xiao and Jennings [6] studied the dynamical properties of the ratio-dependent predator-prey model with nonzero constant harvesting. Gupta et al. [3] studied the stability and bifurcation of the Leslie-Gower predator-prey model in the presence of non-linear harvesting in prey.

The objective of this paper is to study the dynamical properties of a predator-prey system in which prey is not economically useful but predator is so economically useful that we ignore the natural mortality rate of predator and harvesting rate is nonlinear. This type of model is very common in fishery.

We consider the following predator-prey model

$$\frac{dX}{dT} = rX\left(1-\frac{X}{k}\right) - aXY, \frac{dY}{dT} = maXY - \frac{qEY}{m_1E + m_2Y}, \qquad (1)$$

with the initial conditions $X(0) > 0, Y(0) > 0$.

where $X(T)$ is the density of prey at time T, $Y(T)$ is the density of predator at time T, k is the environmental carrying capacity of prey in the absence of predator, a is the attack rate of predators, m is the efficiency with which predators convert consumed prey into offspring, q is the catchability coefficient, E is the effort applied to harvest the predator species, m_1 and m_2 are suitable constant.

To reduce the number of parameters from system (1) we shall use the following non-dimensional parameters

$$X = kx,\ Y = \frac{m_1E}{m_2}y,\ T = \frac{1}{r}t, \alpha = \frac{am_1E}{rm_2},$$

$$\beta = \frac{mak}{r}, h = \frac{q}{rm_1}.$$

Then the system (1) reduces to

$$\frac{dx}{dt} = x(1-x-\alpha y), \frac{dy}{dt} = y\left(\beta x - \frac{h}{1+y}\right), \qquad (2)$$

with the initial conditions $x(0) > 0, y(0) > 0$.

Theorem1. All solutions $(x(t), y(t))$ of the system (2) with the initial conditions are positive.

Proof. From first and second equation of the system (2), we have

$$x(t)=x(0)\exp\left[\int_0^t\left(1-x(s)-\alpha y(s)\right)ds\right]>0,$$

$$y(t)=y(0)\exp\left[\int_0^t\left(\beta y(s)-\frac{h}{1+y(s)}\right)ds\right]>0,$$

respectively. Hence all solutions remain within the first quadrant of the $x-y$ plane starting from an interior point of it. Also the solution trajectories starting from $(x_0,0)$ remain within the positive $x-$axis at all future times and similar result holds for trajectories starting from a point on the positive $y-$axis. Hence, $R_+^2=\{(x,y):x,y\geq 0\}$ is an invariant set.

Equilibrium Points and their Local Stability: $E_{00}=(0,0)$ and $E_{10}=(1,0)$ are the two trivial equilibrium points of the system (2) and the interior equilibrium points are the solution of the equations

$$1-x-\alpha y=0,\beta x-\frac{h}{1+y}=0.$$ (see Fig. 1)

Lemma 1. (a) If $h>\beta(1+\alpha)^2/4\alpha$ then the system (2) has no interior equilibrium points.

(b) If $h=\beta(1+\alpha)^2/4\alpha$ then the system (2) has a unique interior equilibrium point $E_0=(x_0,y_0)$ where $x_0=\frac{1+\alpha}{2}$ and $y_0=\frac{1-\alpha}{2\alpha}$ (C) If $\beta<h<\beta(1+\alpha)^2/4\alpha$ then the system (2) has two interior equilibrium points $E_1=(x_1,y_1)$ and $E_2=(x_2,y_2)$ where

$$x_i=\frac{\beta(1+\alpha)+(-1)^{i+1}\sqrt{\beta^2(1+\alpha)^2-4\alpha\beta h}}{2\beta},$$

$$y_i=\frac{\beta(1-\alpha)+(-1)^{i}\sqrt{\beta^2(1+\alpha)^2-4\alpha\beta h}}{2\alpha\beta},i=1,2$$

(d) If $h<\beta(1+\alpha)^2/4\alpha$ and $h<\beta$ then the system (2) has a unique interior equilibrium point $E_*=(x_*,y_*)=(x_2,y_2)$.

Theorem 2. (a) The equilibrium point E_{00} of the system (2) is always saddle.

(b) The equilibrium point E_{10} of the system (2) is stable if $\beta<h$ and saddle if $\beta>h$.

(c) The equilibrium point E_1 of the system (2) is always saddle.

(d) The equilibrium point E_2 of the system (2) is always stable.

(e) The equilibrium point E_* of the system (2) is always stable.

Proof. The Jacobian matrix of the system (2) at the point $E(x,y)$ is

$$J_E=\begin{bmatrix}1-2x-\alpha y & -\alpha x\\ \beta y & \beta x-\frac{h}{1+y}+\frac{hy}{(1+y)^2}\end{bmatrix}.$$

(a) The eigenvalues of the Jacobian matrix $J_{E_{00}}$ are $\lambda_1=1$ and $\lambda_2=-h.$ Hence the result.

(b) The eigenvalues of the Jacobian matrix $J_{E_{10}}$ are $\lambda_1=-1$ and $\lambda_2=\beta-h.$ Hence the result.

(c) The determinant of the Jacobian matrix J_{E_1} is $Det(J_1)=-\frac{x_1y_1}{1+y_1}\sqrt{\beta^2(1+\alpha)^2-4\alpha\beta h}<0$

Hence the result by Routh-Hurwitz criteria.

(d) The determinant and trace of the Jacobian matrix J_{E_2} is

$$Det(J_2)=\frac{x_1y_1}{1+y_1}\sqrt{\beta^2(1+\alpha)^2-4\alpha\beta h}>0,$$

$$Tr(J_2)=-\frac{x_2}{1+y_2}(1+(1-\beta)y_2)<0.$$

Hence the result by Routh-Hurwitz criteria. (e) Same as in (d).

Saddle-Node Bifurcation: If $h=\frac{\beta(1+\alpha)^2}{4\alpha}$ then the two interior equilibrium points $E_1=(x_1,y_1)$ and $E_2=(x_2,y_2)$ coincide to each other and results a unique interior equilibrium point $E_0=(x_0,y_0)$ where $x_0=\frac{1+\alpha}{2}$, $y_0=\frac{1-\alpha}{2\alpha}$. Thus the number of interior equilibrium points changes from two to zero when the parameter moves from right to left through the critical value $h=h^{[SN]}=\frac{\beta(1+\alpha)^2}{4\alpha}$. Thus there is a chance of bifurcation at the equilibrium point $E_0=(x_0,y_0)$.

Theorem 3. The system (2) undergoes saddle-node bifurcation at the unique interior equilibrium point $E_0=(x_0,y_0)$ if $h=\beta(1+\alpha)^2/4\alpha$.

Proof. To prove this result we shall use Sotomayor's theorem [5]. The determinant of the Jacobian matrix J_{E_0} of the system (2) at the point $E_0=(x_0,y_0)$ is zero and trace is negative. Thus one of the eigenvalues of the Jacobian matrix is zero while second is negative. Let V and W are the eigenvectors of the Jacobian matrix J_{E_0} and $J_{E_0}^T$ respectively and given by

$$V=\begin{bmatrix}-\alpha & 1\end{bmatrix}^T,\ W=\begin{bmatrix}\dfrac{\beta(1-\alpha)}{\alpha(1+\alpha)} & 1\end{bmatrix}^T.$$

Also we have

$$G_h\left(x_0,y_0,h^{[SN]}\right)=\begin{bmatrix}0 & -y_0/(1+y_0)\end{bmatrix}^T$$

and

$$D^2G_h\left(x_0,y_0,h^{[SN]}\right)=\begin{bmatrix}0 & -2\alpha\beta\dfrac{1-\alpha}{1+\alpha}\end{bmatrix}^T.$$

Thus we have

$$W^T.G_h\left(x_0,y_0,h^{[SN]}\right)=-y_0/(1+y_0)\neq 0,$$ and

$$W^T.D^2G_h\left(x_0,y_0,h^{[SN]}\right)(V\quad V)=-2\alpha\beta\frac{1-\alpha}{1+\alpha}.$$

which in nonzero because $y_0=\dfrac{1-\alpha}{2\alpha}\neq 0.$

Hence by Sotomayor's Theorem system (2) undergoes to saddle-node bifurcation.

Biological Interpretation: The value $h^{[SN]}$ is biologically important and known as Maximum Sustainable Yield (MSY). Xiao [1995] showed that if the harvesting parameter satisfies $h>h^{[SN]}$ then the harvesting species goes to extinct and hence system will collapse and if the harvesting parameter satisfies $0<h<h^{[SN]}$ then the harvesting species do not go to extinct and hence system will exist.

Bionomic Equilibrium: An equilibrium point of system (2) is said to be Bionomic equilibrium point if it is Biological equilibrium as well as economical equilibrium i.e. the positive solution of the equations $\dfrac{dx}{dt}=\dfrac{dy}{dt}=\pi(X,Y,E)=0.$

The Economic rent

$$\pi(X,Y,E)=\left(\frac{pqY}{m_1E+m_2y}-c\right)E,$$

where p is the cost price of per unit biomass of the predator species and c is the constant harvesting cost per unit effort.

Thus by definition, we get the bionomic equilibrium $(X_\infty,Y_\infty,E_\infty)=$

$$\left(\frac{q(pq-m_2)}{am\left(m_1pq+m_1m_2(1+c)\right)},\frac{cm_1E_\infty}{pq-m_2},\frac{r(k-x_\infty)(pq-m_2c)}{acm_1}\right)$$

provided $pq>m_2c$ and $pq>m_2$.

Numerical Simulations: 1. $\alpha=0.5,\ \beta=0.185$ then $h=h^{[SN]}=0.208125$ and thus the system (2) has a unique interior equilibrium point $E_0=(x_0,y_0)=(0.75,0.50)$ which is saddle-node (see Fig. 2) and system undergoes a saddle node bifurcation.(see Fig. 3). If $h=0.2$ then the system has two interior equilibrium points $E_1=(x_1,y_1)=(0.898,0.204)$, $E_2=(x_2,y_2)=(0.612,0.796)$. E_1 is always a saddle point while E_2 is always a stable point (see Fig. 4). If $h=0.22$ then the system has no interior equilibrium point and the axial equilibrium point $E_1=(1,0)$ is the globally asymptotically stable (see Fig. 5).

2. $\alpha=0.8,\beta=0.6,h=0.58$ then the system has a unique interior equilibrium point $E_*=(x_*,y_*)=(0.7085,0.3643)$ which is always stable (globally stable) (see Fig. 6).

Results: In this paper, we have analyzed a predator-prey model with nonlinear harvesting rate and zero natural death rates of predator species. We have shown that the number of interior equilibrium points of the system depends upon the parameter h and have either zero or one or two interior equilibria through saddle node bifurcation. Apart from discussing the stability of each equilibrium point, we have also shown that for some parametric conditions system shows bistability and solution trajectories depends upon the initial values. In figure 7 we have sketch the graph between species (prey and predator) and time t for a trajectory of Figure 6 and showed how predator and prey species exist. By using Sotomayor's theorem we have shown that if h is taken as bifurcation parameter then the system undergoes a saddle-node bifurcation.

We have also found the range in which both the species coexist as positive. The bionomic equilibrium point for the system is also found.

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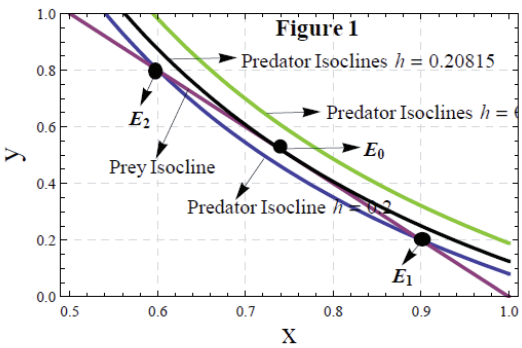

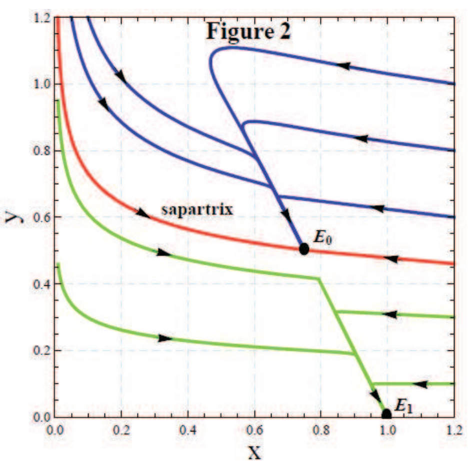

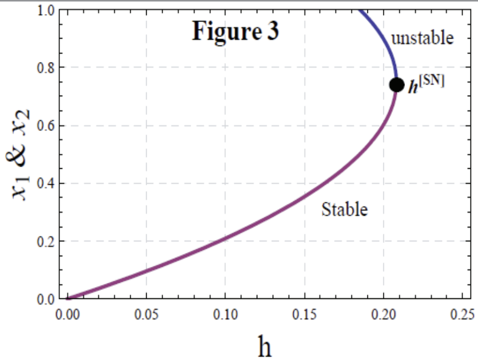

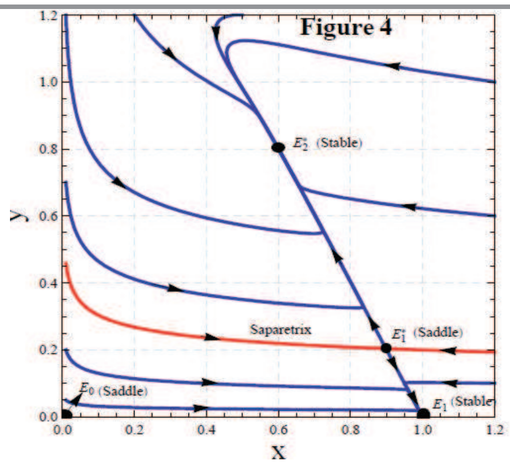

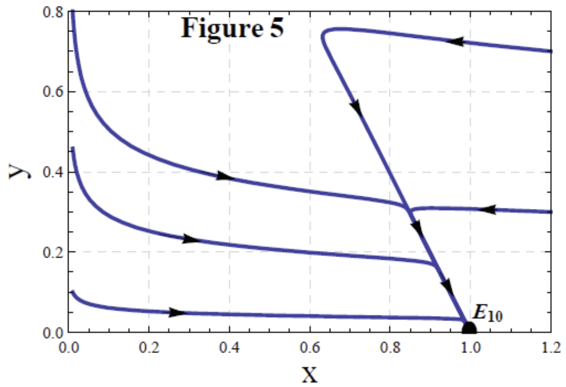

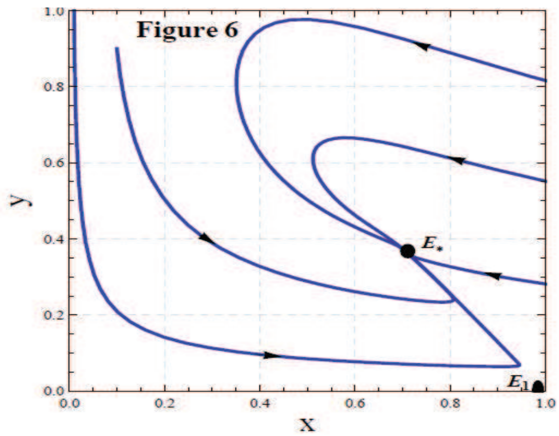

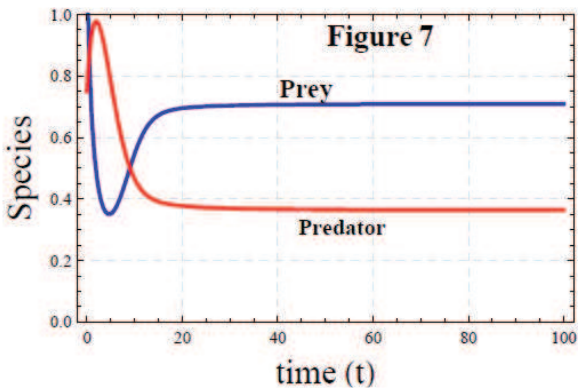

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