SOME COMMON FIXED POINT THEOREMS IN 2-METRIC SPACE FOR RATIONAL EXPRESSION

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Abstract : In this present paper we establish a fixed and common fixed point theorems in complete 2-metric space for new symmetric rational expression. Our result generalize and unify some well known results.

Key Words: common fixed point, complete 2-metric space, rational expression.

Introduction & Preliminaries

The concept of 2-metric space has been investigated by Gahler [5,6]. A 2-metric on X is a mapping d from $X \times X \times X$ to the set of real numbers that satisfies the following condition:

M.1 For two distinct point x, y there is a point Z such that $d(x,y,z) \neq 0$ and $d(x,y,z) = 0$ if at least two of the three points are equal.

M.2 $d(x,y,z) = d(x,z,y) = d(y,z,x)$ for all $x, y, z \ in\ X$.

M.3 $d(x,y,z) \leq d(x,y,u) + d(x,u,z) + d(u,y,z)$ for all $x, y, z, u \ in\ X$.

The pair (X,d) is called a 2-metric space, Now we are introducting some fixed and common fixed point theorems in 2-metric space. This theory was introduced after [5] by , Iseki [8], Sharma and Sharma and Iseki [14], Singh and Norris, [16].

Definition 1.1

A sequence $\{x_n\}$ in a 2-metric space (X,d) is said to be convergent at X if $lim_{n\to\infty} d(x_n, x, z) = 0$ for all z in X.

Definition 1.2

A sequence $\{x_n\}$ in a 2-metric space(X,d) is said to be Cauchy sequence if $lim_{m,n\to\infty} d(x_n, x, z) = 0$ for all z in X.

Definition 1.3

A 2-metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Examples 1.4

Let R^2 be the Euclidean space. Let $d(x,y,z)$ denote the area of the triangle formed by joining the three point $x, y, z \in R^2$. Then (R^2, d) is a 2-metric space and $d(x,y,z) = 0$ for any three distinct points $x, y, z \in R^2$ lying on the same straight line.

2.0 Main Result :-

Theorem 2.1 Let T be a continuous self mapping defined on a complete 2-metric space X, further T, Satisfies the following conditions

$$d(Tx,Ty,u) \leq \alpha \frac{d^2(y,Tx,u)}{d(x,Tx,u)+d(x,y,u)} + \beta \frac{d^2(x,Ty,u)}{d(x,Ty,u)+d(x,y,u)} + \gamma \frac{d^2(y,Ty,u)}{d(y,Ty,u)+d(x,y,u)}$$
$$+ \delta[d(x,Tx,u)+d(y,Ty,u)] + \eta[d(x,Ty,u)+d(y,Tx,u)] + \mu[d(x,y,u)] \ldots\ldots 2.1.a$$

Here $u > 0$ is real and $\forall\, x, y \in X$ with $x \neq y$ with $d(x,y,u) \neq 0$ and

$\alpha + \beta + 2\gamma + 2\delta + 2\eta + \mu < 1$ where

$\alpha, \beta, \gamma, \delta, \eta, \mu \in [0,1)$ then T has unique fixed point of X.

Proof:- Let x_0 be an arbitrary point in X and we define

$x_{n+1} = Tx_n$ and $x_n = Tx_{n-1}$. If Where n is a +ve integer. If $x_n = x_{n+1}$ for some n, then x_n is a fixed point of T. Taking $x_n \neq x_{n+1}$ for all n, then

$$d(x_{n+1},x_n,u) = d(Tx_n,Tx_{n-1},u)$$
$$\leq \alpha \frac{d^2(x_{n-1},Tx_n,u)}{d(x_n,Tx_n,u)+d(x_n,x_{n-1},u)} + \beta \frac{d^2(x_n,Tx_{n-1},u)}{d(x_n,Tx_{n-1},u)+d(x_n,x_{n-1},u)} + \gamma \frac{d^2(x_{n-1},Tx_{n-1},u)}{d(x_{n-1},Tx_{n-1},u)+d(x_n,x_{n-1},u)} +$$
$$\delta[d(x_n,Tx_n,u) + d(x_{n-1},Tx_{n-1},u)] + \eta[d(x_n,Tx_{n-1},u) + d(x_{n-1},Tx_n,u)] + \mu[d(x_n,x_{n-1},u)]$$
$$= \alpha \frac{d^2(x_{n-1},x_{n+1},u)}{d(x_n,x_{n+1},u)+d(x_n,x_{n-1},u)} + \beta \frac{d^2(x_n,x_n,u)}{d(x_n,x_n,u)+d(x_n,x_{n-1},u)}$$
$$+ \gamma \frac{d^2(x_{n-1},x_n,u)}{d(x_{n-1},x_n,u)+d(x_n,x_{n-1},u)}$$

$$+\delta[d(x_n, x_{n+1}, u) + d(x_{n-1,}x_n, u)] + \eta\,[d(x_n, x_n, u) + d\,(x_{n-1}, x_{n+1}, u)] + \mu[d(x_n, x_{n-1}, u)]$$

$$\leq \left(\alpha + \frac{\gamma}{2} + \delta + \eta + \mu\right) d\,(x_{n-1}, x_n, u) + \left(\alpha + \delta + \eta\right) d(x_n, x_{n+1}, u)$$

$$d(x_n, x_{n+1}, u) \leq \frac{\alpha + \frac{\gamma}{2} + \delta + \eta + \mu}{1-\alpha-\delta-\eta} d(x_{n-1}, x_n, u)$$

$$d(x_n, x_{n+1}, u) \leq S\, d(x_{n-1}, x_n, u)$$

where $S = \frac{\alpha + \frac{\gamma}{2} + \delta + \eta + \mu}{1-\alpha-\delta-\eta} < 1$

Since $\alpha + \beta + 2\gamma + 2\delta + \eta < 1$

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$$d\,(x_n, x_{n+1}, u) \leq S^n\,(x_0, x_1, u)$$

By the triangle inequality we have m > n

$$d(x_n, x_m, u) \leq d(x_n, x_{n+1}, u) + d(x_{n+1}, x_{n+2}, u) + \ldots\ldots + d(x_{m-1}, x_m, u)$$

$$\leq (S^n + S^{n+1} + S^{n+2} + \ldots. + S^{m-1})\, d(x_0, x_1, u)$$

Therefore

$$d\,(x_n, x_m, u) \leq \frac{S^n}{1-S}\, d\,(x_0, Tx_0, u)$$

$$\to 0\ as\ m, n \to \infty$$

so $\{x_n\}$ is cauchy sequence in X. So by completeness of X, there is a point $v \in X$ such that $x_n \to v$ as $n \to \infty$ further the continuity of T in X implies

$T(v) = T\,[\lim_{n\to\infty} x_n] = lim_{n\to\infty}\, Tx_n = lim_{n\to\infty}\, x_{n+1} = v$

so v is fixed point of T in X.

Uniqueness:-

Suppose there is any other point w in X when $w \neq v$ i.e. $T(w) = w$, then

$$d(v, w, u) = d(Tv, Tw, u)$$

$$\leq \alpha \frac{d^2(w, Tv, u)}{d(v, Tv, u) + d(v, w, u)} + \beta \frac{d^2(v, Tw, u)}{d(v, Tw, u) + d(v, w, u)} + \gamma \frac{d^2(w, Tw, u)}{d(w, Tw, u) + d(v, w, u)} + \delta[d(v, Tv, u) +$$
$$d(w, Tw, u)] + \eta\,[d(v, Tw, u) + d(w, Tv, u)] + \mu d(v, w, u)$$

$$\leq \left(\alpha + \frac{\beta}{2} + 2\eta + \mu\right) d(v, w, u)$$

$$d(v, w, u) \leq \left(\alpha + \frac{\beta}{2} + 2\eta + \mu\right) d(v, w, u)$$

which is contradiction because $\alpha + \frac{\beta}{2} + 2\eta + \mu < 1$ so $v = w$.

Hence w is also unique fixed point in X.

Theorem 2.2 Let T be a self mapping defined on a complete 2-metric space $\{x, d\}$ $s.t.$ $(2.1.a)$ holds. If for some positive integer p, T^p is continuous then T has a unique fixed point.

Proof:- We define a sequence as in theorem (2.1.a) since it converges to some point v in X. Therefore its subsequence $\{x_{ns}\}$ where $\{ns = np\}$ also converges to v. Also

$$T^p(v) = T^p\,[lim_{s\to\infty} x_{ns}] = lim_{s\to\infty} T^p(x_{ns}) = lim_{n\to\infty}\, x_{ns+1} = v$$

$T^p(v) = v$, so v is fixed point of T^p.

Now we show that $Tv = v$.

Let m be the smallest positive integers such that $T^m(v) = v$,

$T^n(v) \neq v$ for n=1, 2, 3 ....m -1. So choose $T^m v = v$ put $x = v, \&\ y = T^{m-1}v$

$$d(Tv, v, u) = d\,(Tv, T\,(T^{m-1}v), u)$$

$$\le \alpha \frac{d^2(T^{m-1}v,Tv,u)}{d(v,Tv,u)+d(v,T^{m-1}v,u)} + \beta \frac{d^2(v,T(T^{m-1}v),u)}{d(v,T(T^{m-1}v),u)+d(v,T^{m-1}v,u)}$$

$$+\gamma \frac{d^2(T^{m-1}v),T(T^{m-1}v),u)}{d(T^{m-1}v,T(T^{m-1}v),u)+d(v,T^{m-1}v,u)} + \delta[d(v,Tv,u)+d(T^{m-1}v,T(T^{m-1}v),u)]$$

$$+\eta\,[d(v,T(T^{m-1}v),u)+d(T^{m-1}v,Tv,u)] + \mu[d(v,T^{m-1}v,u)]$$

$$= \Big(\alpha+\gamma+\delta+\eta+\mu\Big)d(T^{m-1}v,v,u) + \Big(\alpha+\delta+\eta\Big)d(v,Tv,u)$$

$$d(Tv,v,u) \le \frac{\alpha+\gamma+\delta+\eta+\mu}{1-\alpha-\delta-\eta} d\,(T^{m-1}v,v,u)$$

$d(Tv,v,u) \le K'\,d(T^{m-1}v,v,u)$, where $K' < 1$
on continuous this process we get $d(Tv,v,u) \le (K')^m d(v,Tv,u)$. which is contradiction because $K' < 1$. So v is fixed point of T.

Theorem 2.3 Let T be a self map, defined on a complete 2-metric space X, s.t. for some positive integer X satisfies the condition.

$$d(T^m(x),T^m(y),u) \le \alpha \frac{d^2(y,T^m x,u)}{d(x,T^m x,u)+d(x,y,u)} + \beta \frac{d^2(x,T^m y,u)}{d(x,T^m y,u)+d(x,y,u)} + \gamma \frac{d^2(y,T^m y,u)}{d(y,T^m y,u)+d(x,y,u)}$$

$$+\delta[d(x,T^m x,u)+d(y,T^m y,u] + \eta\,[d(x,T^m y,u)+d(y,T^m x,u)] + \mu d(x,y,u)$$

$\forall\, x,y \in X$, $x \ne y$ with $d(x,y,u) \ne 0$ and $\alpha,\beta,\gamma,\delta,\eta,\mu \in [0,1)$ *and* $u > 0$ is real with $\alpha+\beta+2\gamma+2\delta+2\eta+\mu < 1$. Then T has unique fixed point in X.

Proof : From the given condition of theorem we assume that T^m has
$T^m(v) = v$ unique fixed point v i.e. again $T(v) = T(T^m v) = T^{m+1}v = v$
We conclude that T is also a fixed point of T^m & T^m has unique fixed point v. So v is unique fixed point of T.

Theorem 2.4 Let S & T be two self maps, defined on a complete 2-metric space X, further T, Satisfies the following conditions

$$d\,(Sx,Ty,u) \le \alpha \frac{d^2(y,Sx,u)}{d(x,Sx,u)+d(x,y,u)} + \beta \frac{d^2\,(x,Ty,u)}{d(x,Ty,u)+d(x,y,u)} + \gamma \frac{d^2(y,Ty,u)}{d(y,Ty,u)+d(x,y,u)}\, \delta[d(x,Sx,u)+ d(y,Ty,u)] + \eta\,[d(x,Ty,u)+d(y,Sx,u)] + \mu\,[d(x,y,u)] \quad \ldots\ldots 2.4.a$$

$\forall\, x,y \in X$ with $x \ne y, u > 0$ is real with $d(x,y,u) \ne 0$ with $\alpha+\beta+3\gamma+2\delta+2\eta+\mu < 1$ where $\alpha,\beta,\gamma,\delta,\eta,\mu \in [0,1)$ then S & T are continuous on X and $\exists$ an $x_0 \in X$, s.t. in the squence $\{x_n\}$ where.

$$x_n = \begin{cases} Sx_{n-1} & \textit{where n is even} \\ Tx_{n-1} & \textit{where n is odd} \end{cases}$$

Then S & T have unique common fixed point in X.

Proof : we choose
$x_{2n+1} = Sx_{2n}$ *and* $x_{2n} = Tx_{2n-1}$

$$d(x_{2n+1},x_{2n},u) = d(Sx_{2n},Tx_{2n-1},u)$$

$$= \alpha \frac{d^2(x_{2n-1},x_{2n+1},u)}{d\,(x_{2n},x_{2n+1},u)+d(x_{2n},x_{2n-1},u)} + \beta \frac{d^2(x_{2n},x_{2n},u)}{d(x_{2n},x_{2n},u)+d(x_{2n},x_{2n-1},u)}$$
$$+\gamma \frac{d^2(x_{2n-1},x_{2n},u)}{d(x_{2n-1},x_{2n},u)+d(x_{2n},x_{2n-1},u)} + \delta\big[d(x_{2n},x_{2n+1},u)+d\big(x_{2n-1},x_{2n},u\big)\big]$$
$$+\eta\,[d(x_{2n},x_{2n},u)+d\,(x_{2n-1},x_{2n+1},u)] + \mu[d(x_{2n},x_{2n-1},u)]$$

$$\leq \alpha d(x_{2n-1}, x_{2n+1}, u) + \frac{\gamma}{2} d(x_{2n-1}, x_{2n}, u) + \delta\left[d(x_{2n}, x_{2n+1}, u) + d\left(x_{2n-1,} x_{2n}, u\right)\right] + \eta\, [d\,(x_{2n-1}, x_{2n+1}, u)] + \mu[d(x_{2n}, x_{2n-1}, u)]$$

$$\leq \left(\alpha + \frac{\gamma}{2} + \delta + \eta + \mu\right) d\,(x_{2n-1}, x_{2n}, u) + \left(\alpha + \delta + \eta\right) d(x_{2n}, x_{2n+1}, u)$$

$$d(x_{2n+1}, x_{2n}, u) \leq \frac{\alpha + \frac{\gamma}{2} + \delta + \eta + \mu}{1 - \alpha - \delta - \eta} d(x_{2n-1}, x_{2n}, u)$$

$$d(x_{2n}, x_{2n+1}, u) \leq K\, d(x_{2n-1}, x_{2n}, u)$$

where $K = \frac{\alpha + \frac{\gamma}{2} + \delta + \eta + \mu}{1 - \alpha - \delta - \eta} < 1$

because $\alpha + \beta + 3\gamma + 2\delta + 2\eta + \mu < 1$

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$d\,(x_{2n+1}, x_{2n}, u) \leq K^{2n}\, d(x_0, x_1, u)$

$d\,(x_{2n+1}, x_{2n+2}, u) \leq K^{2n+1}\, d(x_0, x_1, u)$

So $\{x_n\}$ *is a Cauchy Sequence*

By the completeness of $X, \{x_n\}$ Converges to a point X. Suppose $\{x_n\} \rightarrow p$ and then the sub sequence $\{x_{nl}\}$ also converges to p.

now to $ST(p) = ST\,(lim_{l\rightarrow\infty} x_{nl}) = (lim_{l\rightarrow\infty} x_{nl+1}) = p$.

now be will prove that $TP = p$.

Suppose that $TP \neq p$. then

then $d(p, Tp, u) = d\,(ST(p), T(p), u)$

$$\leq \alpha \frac{d^2(P, ST(p), u)}{d\,(T(p), ST(p), u) + d(T(p), p, u)} + \beta \frac{d^2(T(p), T(p), u)}{d(T(p), T(p), u) + d(T(p), p, u)} + \gamma \frac{d^2(p, T(p), u)}{d(p, T(p), u) + d(T(p), p, u)}$$

$$+\delta[d((T(p), ST(p), u) + d(p, T(p), u)] + \eta\,[d(T(p), T(p), u) + d\,((p, ST(p), u] + \mu\, d((T(p), p, u)$$

$d\,(p, T(p), u), \leq \left(\frac{\gamma}{2} + 2\delta + \mu\right) d(T(p), p, u)$, where $\frac{\gamma}{2} + 2\delta + \mu < 1$

So, Which is contradiction

Therefore $T(p) = p$.

Now $ST(p) = S\,[T(p)] = S(p) = p$.

This is common fixed point of S & T in X.

Uniqueness : If possible let $z\ (where\ z \neq p) b\ and\ another\ comman\ fixed\ point\ of\ S$ & T.

$Therefore\ S(z) = T(z) = z$

Now $d\,(p, z, u) = d(Sp, Tz, u)$

$$\leq \alpha \frac{d^2(z, Sp, u)}{d\,(p, Sp, u) + d(p, z, u)} + \beta \frac{d^2(p, Tz, u)}{d(p, Tz, u) + d(p, z, u)} + \gamma \frac{d^2(z, Tz, u)}{d(z, Tz, u) + d(p, z, u)}$$

$$+\delta[d(p, Sp, u) + d(z, Tz, u)] + \eta\,[d(p, Tz, u) + d\,(z, Sp, u)] + \mu\, d(p, z, u)$$

$$d\,(p, z, u) \leq \left(\alpha + \frac{\beta}{2} + 2\eta + \mu\right) d(z, p, u)$$

So $p = z$, Since $\left(\alpha + \frac{\beta}{2} + 2\eta + \mu\right) < 1$

Hence z a unique common fixed point in .

Theorem 2.5

Let S, G & T be three self mapping defined on a complete 2-metric space X satisfying the condition

$d\,(SG(x), TG(y), u) \leq \alpha \frac{d^2(y,SG(x),u)}{d(x,SG(x),u)+d(x,y,u)} + \beta \frac{d^2\,(x,TG(y),u)}{d(x,TG(y),u)+d(x,y,u)} + \gamma \frac{d^2(y,TG(y),u)}{d(y,TG(y),u)+d(x,y,u)} +$
$\delta[d(x, SG(x), u) + d(y, TG(y), u)] + \eta\,[d(\,x, TG(y), u) + d(y, SG(x), u)] +$
$\mu\,[d(x, y, u)]$ ... ... .2.5. a

$\forall\, x, y \in X$ with $x \neq y\,, u > 0$ is real with $d(x, y) \neq 0$ and $\alpha + \beta + 3\gamma + 2\delta + 2\eta + \mu < 1$ where $\alpha, \beta, \gamma, \delta, \eta\,, \mu \in [0,1)$ then S, F & T have unique common fixed point in X.

Proof:

Choose $x_{2n+2} = SG(x_{2n+1})$ and $x_{2n+1} = TG(x_{2n})$

$$d(x_{2n+2}, x_{2n+1}, u) = d(SG(x_{2n+1}), TG(x_{2n}), u)$$

$$\leq \alpha \frac{d^2(x_{2n}, SGx_{2n+1}, u)}{d\,(x_{2n+1}, SGx_{2n+1}, u) + d(x_{2n+1}, x_{2n}, u)} + \beta \frac{d^2(x_{2n+1}, TGx_{2n}, u)}{d(x_{2n+1}, TGx_{2n}, u) + d(x_{2n+1}, x_{2n}, u)} + \gamma \frac{d^2(x_{2n}, TGx_{2n}, u)}{d(x_{2n}, TGx_{2n}, u) + d(x_{2n+1}, x_{2n}, u)} +$$

$$\delta\left[d(x_{2n+1}, SGx_{2n+1}, u\,) + d\left(x_{2n}, TGx_{2n}, u\right)\right] + \eta\,[d(x_{2n+1}, TGx_{2n}, u) + d\,(x_{2n}, SGx_{2n+1}, u)] + \mu[d(x_{2n+1}, x_{2n}, u)]$$

$$\leq \alpha \frac{d^2(x_{2n}, x_{2n+2}, u)}{d\,(x_{2n+1}, x_{2n+2}, u) + d(x_{2n+1}, x_{2n}, u)} + \beta \frac{d^2(x_{2n+1}, x_{2n+1}, u)}{d(x_{2n+1}, x_{2n+1}, u) + d(x_{2n+1}, x_{2n}, u)} + \gamma \frac{d^2(x_{2n}, x_{2n+1}, u)}{d(x_{2n}, x_{2n+1}, u) + d(x_{2n+1}, x_{2n}, u)} +$$

$$\delta\left[d(x_{2n+1}, x_{2n+2}, u\,) + d\left(x_{2n}, x_{2n+1}, u\right)\right] + \eta\,[d(x_{2n+1}, x_{2n+1}, u) + d\,(x_{2n}, x_{2n+2}, u)] + \mu[d(x_{2n+1}, x_{2n}, u)]$$

$$d(x_{2n+2}, x_{2n+1}, u) \leq \left(\alpha + \frac{\gamma}{2} + \delta + \eta + \mu\right) d\,(x_{2n}, x_{2n+1}, u) + \left(\alpha + \delta + \eta\right) d(x_{2n+1}, x_{2n+2}, u)$$

$$d(x_{2n+2}, x_{2n+1}, u) \leq \frac{\alpha + \frac{\gamma}{2} + \delta + \eta + \mu}{1 - \alpha - \delta - \eta} d(x_{2n}, x_{2n+1}, u)$$

$$d(x_{2n+2}, x_{2n+1}, u) \leq K\, d(x_{2n}, x_{2n+1}, u)$$

where $K = \frac{\alpha + \frac{\gamma}{2} + \delta + \eta + \mu}{1 - \alpha - \delta - \eta} < 1$

because $\alpha + \beta + 3\gamma + 2\delta + 2\eta + \mu < 1$

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$d\,(x_{2n+2}, x_{2n+1}, u) \leq K^{2n}\,(x_0, x_1, u)$

Therefore $d\,(x_{2n+2}, x_{2n+1}, u) \leq K^{2n+1}\,(x_0, x_1, u)$

So $\{x_n\}$ *is a Cauchy Sequence*

By the completeness of X, suppose $\{x_n\} \to p$ and then subsequence $\{x_{nl}\}$ also converge to p.

Now if we assume, $p \neq TG(p)$, *then* $d(p, TG\,(p), u) > 0$

Therefore

$d(p, TG(p), u) \leq d(p, x_{2n+2}, u) + d(x_{2n+2}, TG\,(p), u) \leq d(p, x_{2n+2}, u) + d(SG(x_{2n+1}), TG\,(p), u)$

$$\leq d(p, x_{2n+2}, u) + \alpha \frac{d^2(p, SGx_{2n+1}, u)}{d\,(x_{2n+1}, SGx_{2n+1}, u) + d(x_{2n+1,}, p, u)} + \beta \frac{d^2(x_{2n+1}, TG\,(p), u)}{d(x_{2n+1}, TG(p), u) + d(x_{2n+1}, p, u)}$$
$$+ \gamma \frac{d^2(p, TG(p), u)}{d(p, TG(p), u) + d(x_{2n+1}, p, u)} + \delta[d(x_{2n+1}, SGx_{2n+1}, u\,) + d(p, TG\,(p), u)]$$
$$+ \eta\,[d(x_{2n+1}, TG(p), u) + d\,(p, SGx_{2n+1}, u)] + \mu[d(x_{2n+1}, p, u)]$$

$d(p, TG(p), u) \leq (\gamma + \delta) d\,(p, TG(p), u)$.
Now which is contradiction, because
$\alpha + \beta + 3\gamma + 2\delta + 2\eta + \mu < 1$
So, $TG(p) = p$, similarly we $p \neq SG$, we get a contradiction. Hence
Therefore $SG(p) = TG(p) = p$
So, p is common fixed point SG & TG.
So, p is common fixed point of $S, G,$ & T.
Uniqueness:
IF possible, let q (where $q \neq p$) is another common fixed point of SG & TG.
i.e. $SG(q) = TG(q) = q$

$d(p, q, u) = d(SG(p), TG(q), u)$

$\leq \alpha \frac{d^2(q, SG(p), u)}{d\,(p, SG(p), u) + d(p, q,), u} + \beta \frac{d^2(p\,, TG\,(q), u)}{d(p, TG(q), u) + d(p, q, u)} + \gamma \frac{d^2(q, TG(q), u)}{d(q, TG(q), u) + d(p, q, u)} +$

$\delta[d(p, SG(p\,), u) + d(q, TG\,(q), u)] + \eta\,[d(p, TG(q), u) + d\,(q, SG(p), u)] + \mu[d(p, q, u)]$
$d(p, q, u) \leq \left(\alpha + \frac{\beta}{2} + 2\eta + \mu\right) d(p, q, u)$
Which is contradiction because $\alpha + \frac{\beta}{2} + 2\eta + \mu < 1$
So, q is common fixed point of $S\,G$ & TG.
So p & q is common fixed point of SG & PG in X.
Hence, common fixed point is unique.

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