DETERMINISTIC INVENTORY MODEL FOR DETERIORATING ITEMS IN A TWO-WAREHOUSE SYSTEM- AN OVERVIEW

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Abstract: In this paper, we discuss a deterministic model for an item permitting deterioration. Use of a rented warehouse is also made besides the own warehouse. The problem develops with different deterioration rates of items in the two warehouses, involves backlogging demand rate, shortages in the owned warehouse etc. We finally provide a procedure to find the maximum total profit per unit time and end up with an unique optimal replenishment policy.

Keywords: Backlogging rate, Capacity constraint, Deteriorating items, Deterministic Inventory model, Two-warehouse problem.

Introduction: Classical decision-makers have a general assumption is that the organization owns a single warehouse with infinite capacity which is practically infeasible. Because of some diverse reasons like the price discount for bulk purchases or predicted high demand of items or over production etc. the organizations order more items than the capacity of OW. Owing to the limited capacity of the owned warehouse (OW), one or more additional warehouses are required possibly rented warehouse (RW), assumed to be of abundant capacity. It is obvious here that holding cost in RW is higher than that in OW. It would be profitable to consume the goods of RW at the earliest. Thus, the stocks of OW will be released only after the stocks of RW are used up.

Hartely [1] made an early discussion regarding use of two warehouses. Sarma [2] developed a deterministic inventory model with infinite replenishment rate and two levels of storage. Murdeshwar and Sathe [3] extended this model to the case of finite replenishment rate. Dave [4] further discussed the cases of bulk release pattern for both finite and infinite replenishment rates. Assuming the deterioration in both warehouses, Sarma [5] extended his earlier model to the case of infinite replenishment rate with shortages. Pakkala and Achary [6]-[7] extended the two-warehouse inventory model for deteriorating items with finite replenishment rate and shortages with constant demand rate. The ideas of time-varying demand and stock-dependent demand were considered by some authors, such as Goswami and Chaudhuri [8]-[9], Bhunia and Maiti [10], [12], Benkherouf [11], Kar et al. [13] and others. Routaray and Sahu[18] proposed a model for determining the optimal replenishment cycle for the two-warehouse inventory problem under inflation, in which the inventory deteriorates at a constant rate over time and shortages were allowed.

The above articles have not allowed shortages. Zhou [14] presented a multi-warehouse inventory model for non-perishable items with time-varying demand and partial backlogging and the backlogging function was assumed to be dependent on the amount of demand backlogged. Zhou and Yang [17] discussed a two-warehouse inventory model for items with stock-level-dependent demand rate that is in polynomial form. Abad [15], [16] discussed a pricing and lot-sizing problem for a product with a variable rate of deterioration, allowing shortages and partial backlogging. The backlogging rate depends on the time to replenishment. Routaray [20] also discussed the optimal lot-size determination for a two-warehouse problem with deterioration and shortages using net present value

Decision-makers have recognized that besides maximizing profit, customer satisfaction plays an important role for getting and keeping a successful position in a competition. So a proper inventory level should be set based on the relationship between the investment in inventory and the service level. With a lost sale, the customer's demand for the item is lost and replaced by a competitor. It can be considered as the loss of profit on the sales. Moreover, it also includes the cost of losing the customer, loss of goodwill, and establishing a poor service record. Therefore, if we omit the stock-out cost from the total profit, then the profit will be over rated. It is true that the stock-out cost is very difficult to measure. In practice, the stock out cost can be easy to obtain from accounting data. In this paper, we develop a deterministic inventory model for deteriorating items with two warehouses involving variety of constraints.

We assume that the inventory costs (including holding cost and deterioration cost) in RW are higher than those in OW. Moreover, shortages are allowed in the owned warehouse and the backlogging rate of unsatisfied demand is assumed to be a decreasing function of the waiting time. We then prove that the optimal replenishment policy not only exists but is unique too. A numerical example is here given to illustrate the proposed model.

Notations: To develop the mathematical model of inventory replenishment schedule with two warehouses, the notation adopted in this paper is as below:

Assumptions: In addition, the following assumptions are imposed

- Replenishment rate is infinite
- Lead time is zero.
- The time horizon of the inventory system is infinite.
- The owned warehouse (OW) has a fixed capacity of W units; the rented warehouse (RW) has unlimited capacity.

D the demand rate per unit time
A the replenishment cost per order
C the purchasing cost per unit
S the selling price per unit, where S > C
W the capacity of the owned warehouse
Q the ordering quantity per cycle
B the maximum inventory level per cycle
C_{11} the holding cost per unit per unit time in OW
C_{12} the holding cost per unit per unit time in RW, where $C_{12} > C_{11}$
C_2 the shortage cost per unit per unit time
R the opportunity (goodwill) cost per unit
θ the constant deterioration rate
α the deterioration rate in OW, where $0 < \alpha < 1$
β the deterioration rate in OW, where $0 < \beta < 1$
t_w the time at which the inventory level reaches zero in RW
t_1 the time at which the inventory level reaches zero in OW
t_2 the length of period during which shortages are allowed
T the length of the inventory cycle, hence $T = t_1 + t_2$
$I_1(t)$ the level of positive inventory in RW at time t
$I_2(t)$ the level of positive inventory in OW at time t
$I_3(t)$ the level of negative inventory at time t
$P(t_w, t_2)$ the total profit per unit time in the two-warehouse case
$P(t_1, t_2)$ the total profit per unit time under the case without capacity constraint in OW

- The goods of OW are consumed only after consuming the goods kept in RW.
- We assume that the maximum deteriorating quantity for items in OW, $(\theta+\alpha)W$, is less than the demand rate D; that is, $(\theta+\alpha)W < C$.
- The unit inventory costs per unit time in RW are higher than in OW, i.e. $C_{12} + (\theta+\beta)C > C_{11} + (\theta+\alpha)C$
- Shortages are allowed. Unsatisfied demand is backlogged.
- The fraction of shortages backordered is $\frac{1}{1+\delta x}$ where x is the waiting time up to the next replenishment and δ is a positive constant.

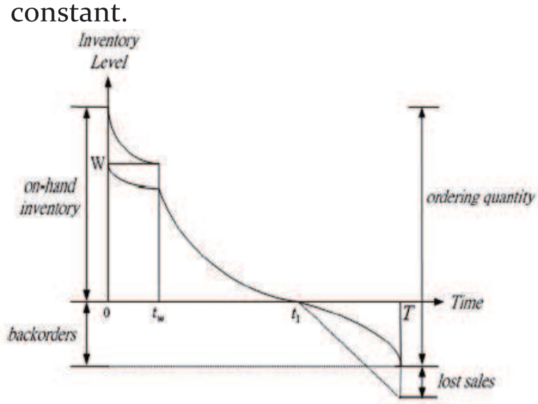

Fig. 1. Graphical representation of a two-warehouse inventory system.

Mathematical formulation: Using above assumptions, the inventory level follows the pattern depicted in Fig. 1. To establish the total relevant profit function, we consider the time intervals separately as $[0,t_w]$, $[t_w,t_1]$ and $[t_w,T]$. During the time interval $[0,t_w]$ the inventory levels are positive at RW and OW. At RW, the inventory is depleted due to the combined effects of demand and deterioration. At OW, the inventory is only depleted by the effect of deterioration. Hence, the inventory level at RW and OW are governed by the following differential equations. Hence, the inventory level at RW and OW are governed by the following differential equations.

$$\frac{dI_1}{dt}+\theta I_1=-D-\beta I_{1,} \quad 0\le t\le t_w,\ I(t_w)=0 \quad (1)$$

$$\frac{dI_2}{dt}+\theta I_2=-\alpha I_2,\ 0\le t\le t_w,\ I(0)=W \quad (2)$$

From equations (1) and (2)

$$I_1=\frac{D}{\theta+\beta}(e^{(\theta+\beta)(t_w-t)}-1),\ 0\le t\le t_w \quad (3)$$

$$I_2=We^{-(\theta+\alpha)t},\quad 0\le t\le t_w \quad (4)$$

During the interval $[t_w,t_1]$ the inventory in OW is depleted due to the combined effects of demand and deterioration.

$$\frac{dI_2}{dt}+\theta I_2=-D-\alpha I_2,\ t_w\le t\le t_1 \quad (5) \qquad I_2=\frac{D}{\theta+\alpha}(e^{(\theta+\alpha))t_1-t)}-1),\ t_w\le t\le t_1 \quad (6)$$

Due to continuity of $I_2(t)$ at point $t=t_w$ from equations (4) and (5) we obtain

$$We^{-(\theta+\alpha)t_w}=\frac{D}{\theta+\alpha}(e^{(\theta+\alpha)(t_1-t_w}-1) \quad (7)$$

This implies $t_1=t_w+\frac{1}{\theta+\alpha}\ln(1+\frac{W(\theta+\alpha)e^{-(\theta+\alpha)t_w}}{D})$ (8)

Here t_1 is a function of t_w. Taking the first order derivative of t_1 with respect to t_w, we have

$$\frac{dt_1}{dt_w}=\frac{1}{1+\frac{\alpha We^{-(\theta+\alpha)t_w}}{D}}<1 \quad (9)$$

Thus, $\frac{dt_1}{dt_w}-1<0$ holds.

Moreover at time t_1, the inventory level reaches zero in OW and shortage occurs. During $[t_1,T]$ the inventory level only depends upon demand and some demand is lost while a fraction $\frac{1}{1+\delta(T-t)}$ of the demand is backlogged when $t\in[t_1,T]$.

$$\frac{dI_3}{dt}=-\frac{D}{1+\delta(T-t)},\ t_1\le t\le T \quad (10)$$

$$I_3(t)=-\frac{D}{\delta}\{\ln(1+\delta(T-t_1)-\ln(1+\delta(T-t))\} \quad (11)$$

The ordering quantity Q over the cycle

$$Q=I_1(0)+I_2(0)-I_3(T)$$
$$=\frac{D}{\theta+\beta}\left(e^{(\theta+\beta)t_w}-1\right)+W+\frac{D}{\delta}\ln(1+\delta t_2) \quad (12)$$

The maximum inventory level per cycle is

$$B=I_1(0)+I_2(0)=\frac{D}{\theta+\beta}\left(e^{(\theta+\beta)t_w}-1\right)+W \quad (13)$$

Ordering cost per cycle =A

Holding cost per cycle in RW $= C_{12}\int_0^{t_w} I_1(t)dt$

$$= C_{12}\frac{D}{(\theta+\beta)^2}\left[\left(e^{(\theta+\beta)t_w}-1\right)-t_w(\theta+\beta)-1\right]$$

Holding cost per cycle in OW $= C_{11}(\int_0^{t_w} I_2(t)dt + \int_{t_w}^{t_1} I_2(t)dt) = \frac{C_{11}}{(\theta+\alpha)}[W - D(t_1 - t_w)]$The shortage cost per cycle $= C_2\int_{t_1}^{T} - I_3(t)dt = C_2 D[\delta t_2 - \ln(1+\delta t_2)]/\delta^2$

Opportunity cost due to lost sales per cycle $= RD\int_{t_1}^{T}\left\{1-\frac{D}{1+\delta(T-t)}\right\}dt = RD\left[\delta t_2 - \ln(1+\delta t_2)\right]/\delta$

The purchase cost per cycle $= CQ = C\left\{\frac{D}{\theta+\beta}\left(e^{(\theta+\beta)t_w}\right)+W+\frac{D}{\delta}\ln(1+\delta t_2)\right\}$ Sales revenue per cycle

$$= S\left[\int_0^{t_1} Ddt + \int_{t_1}^{T}\frac{D}{1+\delta(T-t)}dt\right]$$

$$= \frac{SD}{\delta}\left[\delta t_1 + \ln(1+\delta t_2)\right]$$

Profit function

$$\begin{aligned}\mathrm{P}(\mathrm{t_w},\mathrm{t_2}) = \frac{1}{\mathrm{t_1}+\mathrm{t_2}}&(\text{sales revenue} - \text{ordering cost}\\ &- \text{holding cost-shortage cost}\\ &- \text{opportunity cost} - \text{purchase cost})\end{aligned}$$

$$\begin{aligned}&= D(S-C) - \frac{A}{t_1+t_2} - \frac{C}{t_1+t_2}\left[W + \frac{D}{\theta+\beta}\left(e^{(\theta+\beta)t_w}-1\right) - Dt_1\right]\\ &- \frac{C_{11}}{(\theta+\alpha)(t_1+t_2)}[W - D(t_1-t_w)]\\ &- \frac{C_{12}D}{(\theta+\beta)^2(t_1+t_2)}\left[\left(e^{(\theta+\beta)t_w}-1\right)-(\theta+\beta)t_w - 1\right]\\ &- \frac{D(C_2+\delta(S-C+R)}{(t_1+t_2)\delta^2}\left(\delta t_2 - \ln(1+\delta t_2)\right)\end{aligned}$$

(14)

Differentiating partially with respect to t_w and t_2

$$\begin{aligned}\frac{\partial P(t_w,t_2)}{\partial t_w} &= \left(\frac{1}{(t_1+t_2)(1+(\theta+\alpha)We^{-(\theta+\alpha)t_w}/D}\right)\\ &-\left(D(S-C)-P(t_w,t_2)D\right)\\ &\times\left[\begin{aligned}&(C_{11}+(\theta+\alpha)C)\frac{We^{-(\theta+\alpha)t_w}}{D}\\ &+\frac{(C_{12}+(\theta+\beta)C)(e^{(\theta+\beta)t_w}-1)}{(\theta+\beta)}(1+\frac{(\theta+\alpha)We^{-(\theta+\alpha)t_w}}{D})\end{aligned}\right]\end{aligned} \quad (15)$$

$$\frac{\partial P(t_w,t_2)}{\partial t_2} = \frac{1}{(t_1+t_2)}\left(\begin{aligned}&D(S-C)-P(t_w,t_2)\\ &-\frac{D(C_2+\delta(S-C+R)t_2)}{1+\delta t_2}\end{aligned}\right)(16)$$

The optimal solution of (t_w, t_2) must satisfy $\frac{\partial P(t_w,t_2)}{\partial t_w} = 0$ and $\frac{\partial P(t_w,t_2)}{\partial t_2} = 0$. Solving them we get

$$D(S-C)-P(t_w,t_2)$$
$$= D[(C_{11}+(\theta+\alpha)C)\frac{We^{-(\theta+\alpha)t_w}}{D}$$
$$+\frac{(C_{12}+(\theta+\beta)C)(e^{(\theta+\beta)t_w}-1)}{(\theta+\beta)} \quad (17)$$
$$\times(1+\frac{(\theta+\alpha)We^{-(\theta+\alpha)t_w}}{D})]$$

and $D(S-C)-P(t_w,t_2)=\frac{D(C_2+\delta(S-C+R)t_2)}{1+\delta t_2}$ (18)

respectively. Because both the left hand sides in equation. (17) and (18) are the same, the right hand sides in these equations are equal, that is,

$$\frac{D(C_2+\delta(S-C+R)t_2)}{1+\delta t_2}=\begin{bmatrix}(C_{11}+(\theta+\alpha)C)\frac{We^{-(\theta+\alpha)t_w}}{D}\\+\frac{(C_{12}+(\theta+\beta)C)(e^{(\theta+\beta)t_w}-1)}{(\theta+\beta)}\\(1+\frac{(\theta+\alpha)We^{-(\theta+\alpha)t_w}}{D})\end{bmatrix} \quad (19)$$

Further, we substitute $P(t_w,t_2)$, in (14) into equation (18) and obtain

$$\frac{(C_2+\delta(S-C+R)t_2)(t_1+t_2)}{1+\delta t_2}$$
$$=A+C\left[W+\frac{D}{\theta+\beta}\left(e^{(\theta+\beta)t_w}-1\right)-Dt_1\right]$$
$$+\frac{C_{11}}{(\theta+\alpha)}\left[W-D(t_1-t_w)\right] \quad (20)$$
$$+\frac{C_{12}D}{\left((\theta+\beta)^2\right)}\left[\left(e^{(\theta+\beta)t_w}-1\right)-(\theta+\beta)t_w-1\right]$$
$$+\frac{D(C_2+\delta(S-C+R)}{\delta^2}\left(\delta t_2-\ln(1+\delta t_2)\right)$$

When $t_w>0$,

$$K(t_w)=\begin{bmatrix}\left(C_{11}+(\theta+\alpha)C)\right)\frac{We^{-(\theta+\alpha)t_w}}{D}\\+\frac{\left(C_{12}+(\theta+\beta)C(e^{(\theta+\beta)t_w}-1)\right)}{(\theta+\beta)}\left(1+\frac{((\theta+\alpha)We^{-(\theta+\alpha)t_w}}{D}\right)\end{bmatrix} \quad (21)$$

Lemma 1:

$K(t_w)$ is a continuous and strictly increasing function of $t_w\in[0,\infty)$ and its range is $\left[\frac{W}{D}\left(C_{11}+(\theta+\alpha)C\right),\infty\right)$

Proof: Taking the derivative of $K(t_w)$ with respect to t_w, we have

$$\frac{dK(t_w)}{dt_w}=\left(C_{12}+(\theta+\beta)C\right)\frac{(\theta+\alpha)W\left(e^{-(\theta+\alpha)t_w}\right)}{D}H(t_w)$$

where

$$+\left[\left(C_{12}+(\theta+\beta)C\right)\left(C_{11}+(\theta+\alpha)C\right)\right]\frac{(\theta+\alpha)W\left(e^{-(\theta+\alpha)t_w}\right)}{D}$$

$$H(t_w)=\frac{D}{W}\frac{e^{(\alpha+\beta+2\theta)t_w}}{(\theta+\alpha)}+\frac{e^{-(\theta+\alpha)t_w}}{(\theta+\beta)}(\beta-\alpha),\ t_w\geq 0$$

$$\frac{dH(t_w)}{dt_w}=\frac{D}{W}\frac{(\alpha+\beta+2\theta)}{(\theta+\alpha)}e^{(\alpha+\beta+2\theta)t_w}-e^{(\theta+\beta)t_w}(\beta-\alpha)$$
$$>(\alpha+\beta+2\theta)\left(\frac{D}{(\theta+\alpha)W}-1\right)$$

(by assumption 5)

$H(t_w)$ is a strictly increasing function of $t_w\in[0,1)$, which implies for $t_w>0, H(t_w)>H(0)=\frac{D}{(\theta+\alpha)W}>0$. Thus, from the above result and Assumption 6, we know that $\frac{dK(t_w)}{dt_w}>0,\ t_w>0$.Therefore, $K(t_w)$ is a strictly increasing function of $t_w\in[0,\infty)$. Then the obvious facts are $K(0)=\frac{(C_{11}+(\theta+\alpha)C)W}{D}$ and $\lim_{t_w\to\infty}K(t_w)=\infty$ .Thus the proof.

For any given $t_w\in[0,\infty)$ from equation (19), we define a function

$F(t_2)=(C_2+\delta(S-C+R))\frac{t_2}{1+\delta t_2}-K(t_w),t_2\geq 0$ (22)Then, if

$$\frac{(C_2+\delta(S-C+R))}{\delta}\leq K(0)=\frac{(C_{11}+(\theta+\alpha)C)W}{D}$$

This

$$F(t_2)<\frac{C_2+\delta(S-C+R)}{\delta}-K(t_w)$$
$$<\frac{C_2+\delta(S-C+R)}{\delta}-K(0)$$
$$=\frac{C_2+\delta(S-C+R)}{\delta}-(C_{11}+(\theta+\alpha)C)\frac{W}{D}<0,$$
$$t_2\in[0,\infty)$$

implies, for any given $t_w\in[0,\infty)$, there does not exist a value $t_2\in[0,\infty)$, such that $F(t_2)=0$, i.e. we cannot find a value t_2 which satisfies equation (19). However, for this situation, from equation (15), we get

$$\frac{\partial P(t_w,t_2)}{\partial t_w}=\left(\frac{D}{(t_1+t_2)}\frac{1}{(1+\frac{(\theta+\alpha)We^{-(\theta+\alpha)t_w}}{D}}\right)F(t_2)<0$$ Thus, when

$$\frac{(C_2+\delta(S-C+R))}{\delta}\leq\frac{(C_{11}+(\theta+\alpha)C)W}{D}$$

or equivalently

$$W\geq\frac{D(C_2+\delta(S-C+R))}{\delta(C_{11}+(\theta+\alpha)C)},$$

the maximum value of $P(t_w,t_2)$ occurs at boundary point $t_w^*=0$.

In the special circumstance that $t_w^*=0$, the optimal value of t_1 (denoted by t_1^*) can be obtained by equation (8) and is $t_1^*=\frac{1}{\theta+\alpha}\ln(1+\frac{W(\theta+\alpha)}{D})$. Besides, from equation (13), the maximum inventory level per cycle is B* = W. Then, putting t_w^* and t_2^* into equation (20), the optimal value of

t_2 , should be selected to satisfy

$$D(C_2+\delta(S-C+R))\frac{(t_1^*+t_2)t_2}{1+\delta t_2} = A+\left(C+\frac{C_{11}}{\theta+\alpha}\right)\left(W-Dt_1^*\right) + D(C_2+\delta(S-C+R))\frac{(t_1^*+t_2)t_2}{1+\delta t_2} \quad (23)$$

We want to prove that the value of t_2 which satisfies equation (23) is unique.Let

$$Z(t_2)=D(C_2+\delta(S-C+R))\frac{(t_1^*+t_2)t_2}{1+\delta t_2} - A-\left(C+\frac{C_{11}}{\theta+\alpha}\right)\left(W-Dt_1^*\right) - D(C_2+\delta(S-C+R))\frac{(t_1^*+t_2)t_2}{1+\delta t_2},\ t_2\geq 0$$

$$\frac{dZ(t_2)}{dt_2}=D(C_2+\delta(S-C+R))\frac{(t_1^*+t_2)}{\left(1+\delta t_2\right)^2}>0$$ And

$\lim_{t_2\to\infty} Z(t_2)=\infty$. By using initial value theorem $\exists$ a unique solution $t_2=t_2^*\in\left(0,\infty\right)$ such that $Z(t_2^*)=0$.

Theorem 1. If $W\geq\frac{D(C_2+\delta(S-C+R))}{\delta(C_{11}+(\theta+\alpha)C)}$, then the optimal value of (t_w,t_1,t_2) is given by $t_w^*=0$ and $t_1^*=\frac{1}{\theta+\alpha}\ln(1+\frac{W(\theta+\alpha)}{D})$, t_2 value satisfies equation (23).

Proof: If $W\geq\frac{D(C_2+\delta(S-C+R))}{\delta(C_{11}+(\theta+\alpha)C)}$ then the capacity of OW is sufficient and the maximum inventory level per cycle is $B^*=W$. Besides the optimal inventory cycle is $T^*=t_1^*+t_2^*$. Once the optimal solution $(t_w^*,t_2^*)=(0,t_2^*)$ is obtained, we substitute into equation (12) and (14) together with

$$t_1^*=\frac{1}{\theta+\alpha}\ln(1+\frac{W(\theta+\alpha)}{D})$$

and $Q^*=W+\frac{D\ln(1+\delta t_2^*)}{\delta}$ and $P(0,t_2^*)=D(S-C)-\frac{D(C_2+\delta(S-C+R))t_2^*}{\left(1+\delta t^*_2\right)}$ (24)

$$K(t_w)\geq K(\hat{t_w})=\frac{(C_2+\delta(S-C+R))}{\delta}$$

$$>\frac{(C_2+\delta(S-C+R))}{\delta}-\frac{(C_2+\delta(S-C+R))}{\delta}\frac{1}{\left(1+\delta t_2\right)}$$ Besides, we have

$$=\frac{(C_2+\delta(S-C+R))}{1+\delta t_2}$$

$$F(0)=-K(t_w)<-K(0)=-\frac{W(C_{11}+(\theta+\alpha)C}{D}<0$$

$$\lim_{t_2\to\infty}F(t_2)=\frac{C_2+\delta(S-C+R)}{\delta}-K(t_w) > \frac{C_2+\delta(S-C+R)}{\delta}-K(\hat{t_w})=0$$

Thus, $\exists$ a unique $t_2 \in (0,\infty)$ such that $F(t_2)=0$.Consequently, when $\frac{C_2+\delta(S-C+R)}{\delta} < \frac{W(C_{11}+(\theta+\alpha)C}{D}$ or equivalent,

$$W < \frac{D(C_2+\delta(S-C+R))}{\delta(C_{11}+(\theta+\alpha)C}$$

$\exists$ unique t_2 such that $C_2+\delta(S-C+R)\frac{t_2}{1+\delta t_2} = K(t_w)$ (25)

Implies

$$t_2 = \frac{K(t_w)}{D(C_2+\delta(S-C+R))-\delta K(t_w)} \quad (26)$$

$$\frac{dt_2}{dt_w} = \frac{K(t_w)}{\left[D(C_2+\delta(S-C+R))-\delta K(t_w)\right]^2} > 0 \quad (27)$$

$$t_1^* = t_w^* - \frac{1}{(\theta+\alpha)}\ln\left[1+\frac{(\theta+\alpha)We^{-(\theta+\alpha)t_w}}{D}\right] \quad (28) \qquad t_2^* = \frac{K(t_w^*)}{D(C_2+\delta(S-C+R))-\delta K(t_w^*)} \quad (29)$$

and $T^* = t_1^* + t_2^*$ (30)

Let

$$\begin{aligned} G(t_w) = {} & A + C\left[W + \frac{D}{(\theta+\beta)}(e^{(\theta+\beta)t_w}) - Dt_1\right] \\ & + \left(\frac{C_{11}}{\theta+\alpha}\right)\left(W - D(t_1 - t_w)\right) \\ & + \frac{DC_{12}}{(\theta+\beta)^2}(e^{(\theta+\beta)t_w} - (\theta+\beta)t_w - 1) \quad (31) \\ & + \frac{D(C_2+\delta(S-C+R))}{\delta^2}\left(\delta t_2 - \ln(1+\delta t_2)\right) \\ & - D(C_2+\delta(S-C+R))\frac{(t_1+t_2)t_2}{1+\delta t_2} \end{aligned}$$

$\frac{dG(t_w)}{dt_w} = -D(C_2+\delta(S-C+R))\frac{(t_1+t_2)}{1+\delta t_2}\frac{dt_2}{dt_w} < 0$ Therefore,when

$$t_1^* = t_w^* - \frac{1}{(\theta+\alpha)}\ln\left[1+\frac{(\theta+\alpha)We^{-(\theta+\alpha)t_w}}{D}\right]$$

$$\begin{aligned} & \lim_{t_w \to \hat{t_w}} G(t_w) \\ & = A + \frac{C_{11}+(\theta+\alpha)C}{(\theta+\alpha)}\left[W - \frac{D}{(\theta+\alpha)}\ln\left(1+\frac{(\theta+\alpha)We^{-(\theta+\alpha)\hat{t_w}}}{D}\right)\right] \\ & + \frac{D(C_{12}(\theta+\beta)C)}{(\theta+\beta)^2}\left(e^{(\theta+\beta)\hat{t}_w} - (\theta+\beta)\hat{t}_w - 1\right) \\ & - \frac{D(C_2+\delta(S-C+R))}{\delta}\hat{t_1} + \frac{D(C_2+\delta(S-C+R))}{\delta^2} \\ & - \frac{D(C_2+\delta(S-C+R))}{\delta^2}\lim_{t_w \to \hat{t_w}} \ln(1+\delta t_2) \end{aligned}$$

From equation (26) we obtain, $t_2 \to \infty$ and $t_w \to \hat{t_w}$. Hence, $\lim_{t_w \to \hat{t_w}} G(t_w) = -\infty$ and

$$G(0) = A + \frac{W\left(C_{11} + (\theta+\alpha)C\right)}{\delta} - \left[\frac{D\left(C_{11} + (\theta+\alpha)C\right)}{(\theta+\alpha)^2}\right] \left[\left(1+\frac{(\theta+\alpha)W}{D}\right)\ln\left(1+\frac{(\theta+\alpha)W}{D}\right) - \left(1+\frac{(\theta+\alpha)W}{D}\right)\right] - \left[\frac{D(C_2+\delta(S-C+R))}{\delta^2}\right] \left[\ln\left\{\frac{D(C_2+\delta(S-C+R))}{D(C_2+\delta(S-C+R)) - \delta(C_{11}+(\theta+\alpha)C)}\right\}\right]$$

(32)

and

$$\Delta = A + \frac{W\left(C_{11} + (\theta+\alpha)C\right)}{\delta} - \frac{D\left(C_{11} + (\theta+\alpha)C\right)}{(\theta+\alpha)^2} \times \left[\left(1+\frac{(\theta+\alpha)W}{D}\right)\ln\left(1+\frac{(\theta+\alpha)W}{D}\right) - \left(1+\frac{(\theta+\alpha)W}{D}\right)\right] - \frac{D(C_2+\delta(S-C+R))}{\delta^2} \times \ln\left\{\frac{D(C_2+\delta(S-C+R))}{D(C_2+\delta(S-C+R)) - \delta(C_{11}+(\theta+\alpha)C)}\right\}$$

Lemma 2*:*

For $W < \dfrac{D(C_2+\delta(S-C+R))}{\delta(C_{11}+(\theta+\alpha)C)}$, we have

(a) If $\Delta > 0,$ then the solution $t_w^* \in (0, \hat{t}_w)$ which satisfies equation(20) not only exists but also is unique.

(b) If $\Delta \le 0,$ then the optimal value of t_w is $t_w^* = 0.$

Proof:(a) If $\Delta > 0,$ i.e., $G(0) > 0.$ Since $G(t_w)$ is a strictly decreasing function in $t_w \in (0, \hat{t}_w)$, and $\lim_{t_w \to \hat{t}_w} -G(t_w) < 0,$ by using the Intermediate Value Theorem, there exists a unique solution $t_w^* \in (0, \hat{t}_w)$ such that $G(t_w^*) = 0.$

(b) If $\Delta < 0,$ i.e., $G(0) < 0.$ Hence, for $t_w^* \in [0, \hat{t}_w),$ we know the solution of $G(t_w) = 0$ does not exist. For this situation, from equations (16) and (31), we then obtain that $\dfrac{\partial P(t_w,t_2)}{\partial t_2} = \dfrac{G(t_w)}{(t_1+t_2)^2} < \dfrac{G(0)}{(t_1+t_2)^2} < 0,$ which implies that a smaller value of t_2 causes a higher value of $P(t_w,t_2)$. By using the finding of equation (27), we know that t_2 is a strictly increasing function of t_w; therefore, the maximum value of $P(t_w,t_2)$ occurs at the boundary point $t_w^* = 0.$

For the another case: $\Delta = 0,$ i.e., $G(0) = 0,$ then from the property that $G(t_w)$ is a strictly decreasing function of $t_w^* \in [0, \hat{t}_w),$ we see that $t_w^* = 0$ is the unique solution. This completes the proof.

When $< \dfrac{D(C_2+\delta(S-C+R))}{\delta(C_{11}+(\theta+\alpha)C)}$, Lemma 2(a) shows that $\Delta > 0,$ is the condition for the existence and uniqueness of the solution. On the other hand, even if $W < \dfrac{D(C_2+\delta(S-C+R))}{\delta(C_{11}+(\theta+\alpha)C)}$, Lemma 2(b) reveals that if the ordering cost A, or the unit inventory cost per unit inOW, $C_{11}+(\theta+\alpha)C$ is relatively low so that $\Delta \le 0,$ the inventory model return to the one-warehouse problem.

The unique solution in Lemma 2(a) will be proved to be indeed a global maximum by checking the second order optimality conditions, that is, we have the following main result.

Theorem 2: For $W < \dfrac{D(C_2+\delta(S-C+R))}{\delta(C_{11}+(\theta+\alpha)C)}$, if $\Delta > 0,$ then the point (t_w^*, t_2^*) which satisfies equation (19) and (20) simultaneously is the global maximum of the total profit per unit time.

Proof: If D > 0, then from Lemma 2(a), the solution $t_w^* \in (0, \hat{t_w})$, which satisfies equation (20) not only exists but also is unique. Hence, the value of t_2^* can be determined by equation (29).Moreover, since $0 < \frac{dt_1}{dt_w} < 1$, $\frac{dK(t_w)}{dt_w} < 0$, we obtain $\left[\frac{\partial^2 P(t_w,t_2)}{\partial t_w^2}\right]_{(t_w,t_2)=(t_w^*,t_2^*)} = \left[\frac{-D}{(t_1+t_2)}\frac{dt_1}{dt_w}\frac{dK(t_w)}{dt_w}\right]_{(t_w^*,t_2^*)} < 0$

$\left[\frac{\partial^2 P(t_w,t_2)}{\partial t_2^2}\right]_{(t_w,t_2)=(t_w^*,t_2^*)} = \left[\frac{D(C_2+\delta(S-C+R))}{(t_1+t_2)(1+\delta t_2)^2}\right]_{(t_w^*,t_2^*)} < 0$ and $\left[\frac{\partial^2 P(t_w,t_2)}{\partial t_w \partial t_2}\right]_{(t_w,t_2)=(t_w^*,t_2^*)} = 0$

Thus, the determinant of the Hessian matrix at the stationary point (t_w^*, t_2^*) is

$$H = \left[\frac{\partial^2 P(t_w,t_2)}{\partial t_w^2}\right]_{(t_w^*,t_2^*)} \times \left[\frac{\partial^2 P(t_w,t_2)}{\partial t_2^2}\right]_{(t_w^*,t_2^*)} - \left[\frac{\partial^2 P(t_w,t_2)}{\partial t_w \partial t_2}\right]^2_{(t_w^*,t_2^*)}$$

$$= \left[\frac{D^2(C_2+\delta(S-C+R))}{(t_1+t_2)(1+\delta t_2)^2}\frac{dt_1}{dt_w}\frac{dK(t_w)}{dt_w}\right]_{(t_w^*,t_2^*)} > 0$$

As a result, we can conclude that the stationary point (t_w^*, t_2^*) for our optimization problem is a global maximum. This completes the proof.■

Once the optimal solution (t_w^*, t_2^*) is obtained, we substitute (t_w^*, t_2^*) into Eqs. (12) and (14), the optimal ordering quantity per cycle and the maximum total profit per unit time $P(t_w^*, t_2^*)$ are as follows:

$$Q^* = W + \frac{D(e^{(\theta+\beta)t_w^*}-1)}{\theta+\beta} + \frac{D\ln(1+\delta t_2^*)}{\delta}$$

and

$$P(t_w^*, t_2^*) = D(S-C) - \frac{D(C_2+\delta(S-C+R))t_2^*}{(1+\delta t_2^*)} \quad (33)$$

Inventory problem without capacity constraint in OW

When the OW is so abundant that the RW is not used, the previous model reduces to the OW inventory problem. We remove the capacity constraint of the OW, and hence the total profit per unit time in equation(14) becomes

$$\pi(t_1,t_2) = D(S-C) - \frac{A}{t_1+t_2} - \frac{C_{11}+(\theta+\alpha)C}{(t_1+t_2)(\theta+\alpha)}\left[\frac{D}{\theta+\beta}(e^{(\theta+\beta)t_1}-1) - Dt_1\right] - \frac{D(C_2+\delta(S-C+R))}{(t_1+t_2)\delta^2}[\delta t_2 - \ln(1+\delta t_2)]$$

(34)

Solving the necessary conditions $\frac{\partial \pi(t_1,t_2)}{\partial t_1} = 0$ and $\frac{\partial \pi(t_1,t_2)}{\partial t_2} = 0$ for the maximum value of $\pi(t_1,t_2)$, we get

$$\frac{D(C_2+\delta(S-C+R))t_2}{(t_1+t_2)\delta^2} - \frac{[C_{11}+(\theta+\alpha)C](e^{(\theta+\beta)t_1}-1)}{(\theta+\alpha)} = 0 \quad (35)$$

and

$$A + \frac{C_{11}+(\theta+\alpha)C}{(\theta+\alpha)}\left[\frac{D}{\theta+\alpha}(e^{(\theta+\alpha)t_1}-1) + Dt_1\right] - \frac{D(C_2+\delta(S-C+R))}{\delta^2}[\delta t_2 - \ln(1+\delta t_2)] - D(C_2+\delta(S-C+R))\frac{(t_1+t_2)}{1+\delta t_2} = 0 \quad (36)$$

Equation (35) can be rewritten as

$$t_2 = \frac{[C_{11} + (\theta+\alpha)C](e^{(\theta+\beta)t_1} - 1)}{(\theta+\alpha)[(C_2 + \delta(S - C + R))] - \delta[C_{11} + (\theta+\alpha)C](e^{(\theta+\alpha)t_1} - 1)} \quad (37)$$

We note here t_2 is a function of t_1, and when

$$t_1 \in \left(0, \frac{1}{(\theta+\alpha)}\ln\left(1 + \frac{(\theta+\alpha)[C_2 + \delta(S - C + R)]}{\delta[C_{11} + (\theta+\alpha)C]}\right)\right)$$

$$\begin{aligned} X(t_1) = A &+ \frac{C_{11} + (\theta+\alpha)C}{(\theta+\alpha)}\left[\frac{D}{\theta+\alpha}(e^{(\theta+\alpha)t_1} - 1) - Dt_1\right] \\ &+ \frac{D[C_2 + \delta(S - C + R)]}{\delta^2}[\delta t_2 - \ln(1 + \delta t_2)] \\ &- D[C_2 + \delta(S - C + R)]\frac{(t_1 + t_2)t_2}{(1 + \delta t_2)} \end{aligned} \quad (38)$$

By using similar arguments we can obtain the following results.

Lemma 3: The point $t_1^{**} \in \left(0, \frac{1}{(\theta+\alpha)}\ln\left\{1 + \frac{(\theta+\alpha)[C_2 + \delta(S - C + R)]}{\delta[C_{11} + (\theta+\alpha)C]}\right\}\right)$ which satisfies the equation $X(t_1) = 0$ in equation (38) not only exists but is also unique.

Theorem 3: The point (t_1^{**}, t_2^{**}) which satisfies (35) and (36) simultaneously is the global maximum of the total profit per unit time $\pi(t_1, t_2)$.

$$Q^{**} = \frac{D}{\theta+\alpha}\left(e^{(\theta+\alpha)t_1^{**}} - 1\right) + \frac{D}{\delta}\ln(1 + \delta t_2^{**}),$$

$B^{**} = \frac{D}{\theta+\alpha}\left(e^{(\theta+\alpha)t_1^{**}} - 1\right)$ and $\pi(t_1^{**}, t_2^{**}) = D(S - C) - \frac{D(C_2 + \delta(S - C + R))t_2^{**}}{(1 + \delta t_2^{**})}$ (39)

Case 1. When $W \geq \frac{D(C_2 + \delta(S - C + R))}{\delta[C_{11} + (\theta+\alpha)C]}$,

$t_1^{**} \in \left(0, \frac{1}{(\theta+\alpha)}\ln\left\{1 + \frac{(\theta+\alpha)[C_2 + \delta(S - C + R)]}{\delta[C_{11} + (\theta+\alpha)C]}\right\}\right)$ Consequently $B^{**} = \frac{D}{\theta+\alpha}\left(e^{(\theta+\alpha)t_1^{**}} - 1\right)$

$$< \frac{D}{\theta+\alpha}\left(e^{(\theta+\alpha)\frac{1}{(\theta+\alpha)}\ln\left(1 + \frac{(\theta+\alpha)[C_2 + \delta(S - C + R)]}{\delta[C_{11} + (\theta+\alpha)C]}\right)} - 1\right)$$

$$\leq \frac{D}{\theta+\alpha}\left(e^{(\theta+\alpha)\frac{1}{(\theta+\alpha)}\ln\left(1 + \frac{(\theta+\alpha)W}{D}\right)} - 1\right) = W$$

Case2. When $W < \frac{D(C_2 + \delta(S - C + R))}{\delta(C_{11} + (\theta+\alpha)C)}$

$\frac{1}{(\theta+\alpha)}\ln\left(1 + \frac{(\theta+\alpha)[C_2 + \delta(S - C + R)]}{\delta[C_{11} + (\theta+\alpha)C]}\right) > \frac{1}{(\theta+\alpha)}\ln\left(1 + \frac{(\theta+\alpha)W}{D}\right)$ **(a)** If $\Delta \leq 0$, then

$X\left(\frac{1}{(\theta+\alpha)}\ln\left(1 + \frac{(\theta+\alpha)W}{D}\right)\right) = \Delta \leq 0$ Maximum inventory level per cycle

$$B^{**} = \frac{D}{\theta+\alpha}\left(e^{(\theta+\alpha)t_1^{**}} - 1\right)$$

$$\leq \frac{D}{\theta+\alpha}\left(e^{(\theta+\alpha)\frac{1}{(\theta+\alpha)}\ln\left(1 + \frac{(\theta+\alpha)W}{D}\right)} - 1\right) = W$$

(b) If $\Delta > 0$, then $X\left(\frac{1}{(\theta+\alpha)}\ln\left(1 + \frac{(\theta+\alpha)W}{D}\right)\right) = \Delta > 0.$

Then

$$B^{**} = \frac{D}{\theta+\alpha}\left(e^{(\theta+\alpha)t_1^{**}} - 1\right)$$

$$> \frac{D}{\theta+\alpha}\left(e^{(\theta+\alpha)\frac{1}{(\theta+\alpha)}\ln\left(1+\frac{(\theta+\alpha)W}{D}\right)} - 1\right) = W$$

Special cases

***Case 1:* without shortage**

When $\delta \to \infty$ i.e. the fraction of shortages back ordered is zero.

$$P_1(t_w) \equiv P(t_w,0) = D(S-C) - \frac{A}{t} - \frac{C}{t_1}\left[W + \frac{D}{\theta+\beta}\left(e^{(\theta+\beta)t_w} - 1\right) - Dt_1\right] - \frac{C_{11}}{t_1(\theta+\alpha)}[W - D(t_1 - t_w)] - \frac{DC_{12}}{(\theta+\beta)^2 t_1}\left(\left(e^{(\theta+\beta)t_w}\right) - (\theta+\beta)t_w - 1\right) \quad (40)$$

where t_1 is a function of t_w defined in (8).

The necessary condition to find the optimal solution of $P_1(t_w)$ is

$$\frac{dP_1(t_w)}{dt_w} = \frac{1}{t_1^2\left[\frac{1+(\theta+\alpha)We^{-(\theta+\alpha)t_w}}{D}\right]} \times \left[\begin{array}{l} A + \frac{W[C_{11}+(\theta+\alpha)C]}{(\theta+\alpha)} - \frac{D[C_{11}+(\theta+\alpha)C]}{(\theta+\alpha)} \\ -Dt_1K(t_w) + \left(\frac{D[C_{12}+(\theta+\beta)C]}{(\theta+\beta)^2}\right) \\ \times\left(\left(e^{(\theta+\alpha)t_w}\right) - (\theta+\beta)t_w - 1\right) \end{array}\right] = 0 \quad (41)$$

which implies $A + \frac{W[C_{11}+(\theta+\alpha)C]}{(\theta+\alpha)} - \frac{D[C_{11}+(\theta+\alpha)C](t_1-t_w)}{(\theta+\alpha)} - Dt_1K(t_w) + \frac{D[C_{12}+(\theta+\beta)C]}{(\theta+\beta)^2}\left(\left(e^{(\theta+\alpha)t_w}\right) - (\theta+\beta)t_w - 1\right),\ t_w \ge 0$ Let

$$L(t_w) = A + \frac{W[C_{11}+(\theta+\alpha)C]}{(\theta+\alpha)} - \frac{D[C_{11}+(\theta+\alpha)C](t_1-t_w)}{(\theta+\alpha)} - Dt_1K(t_w) + \frac{D[C_{12}+(\theta+\beta)C]}{(\theta+\beta)^2}\left(\left(e^{(\theta+\alpha)t_w}\right) - (\theta+\beta)t_w - 1\right),\quad t_w \ge 0 \quad (42)$$

$\hat{Q} = W + \frac{D\left(e^{(\theta+\beta)t_w} - 1\right)}{(\theta+\beta)}$ and

$$P_1(\hat{t_w}) = D(S-C) - DK(\hat{t_w}) \quad (43)$$

Case2.Without stock

$$P_2(t_2) \equiv P(0,t_2) = D(S-C) - \frac{A}{t_2} + \frac{D(C_2+\delta(S-C+R))}{t_2\delta^2}(\delta t_2 - \ln(1+\delta t_2)) \quad (44)$$

The necessary condition to find the optimal solution of $P_2(t_2)$ is

$\frac{dP_2(t_2)}{dt_2}=\frac{1}{t_2^2}\left\{A+\frac{D(C_2+\delta(S-C+R))}{\delta^2}\left[\frac{\delta t_2}{1+\delta t_2}-\ln(1+\delta t_2)\right]\right\}=0$ which implies

$$A+\frac{D(C_2+\delta(S-C+R))}{\delta^2}\left[\frac{\delta t_2}{1+\delta t_2}-\ln(1+\delta t_2)\right]=0 \quad (45)$$

Define

$$M(t_2)=A+\frac{D(C_2+\delta(S-C+R))}{\delta^2}\left[\frac{\delta t_2}{1+\delta t_2}-\ln(1+\delta t_2)\right],\ t_2\geq 0 \quad (46)$$

$$P(t_2^{\#})=D(S-C)+\frac{D(C_2+\delta(S-C+R))t_2^{\#}}{1+\delta t_2^{\#}} \quad (47)$$

From equations (42),(46) and (20) we get

$$A=L(t_w)+M(t_2) \quad (48)$$

Theorem 4: For $W<\frac{D(C_2+\delta(S-C+R))}{\delta[C_{11}+(\theta+\alpha)C]}$, if $\Delta>0$, then $P(t_w^*,t_2^*)>\max\left\{P_1(t_w^{\wedge}),P_2(t_2^{\#})\right\}$.

Proof: Because $K(t_w^{\wedge})$ are the optimal solutions of $P_1(t_w)$ in equation (40) and $P(t_w,t_2)$ in equation (14) respectively, from equation (42) and (48) we have

$$L(t_w^{\wedge})=0 \quad (49)$$

And

$$A=L(t_w^*)+M(t_2^*) \quad (50)$$

Equation (50) can be rewritten as $L(t_w^*)=A-M(t_2^*)>A-M(0)=0$ (51)

because $M(t_2)$ is a strictly decreasing function and $M(0)=A$. (52)

Comparing equations (49) and (51), we get $L(t_w^*)>L(t_w^{\wedge})$. Recall that $L(t_w)$ is a strictly decreasing function in $t_w\in[0,\infty]$, equation (52) implies $t_w^{\wedge}>t_w^*$. Then from equations (33) and (43) we obtain

$$\begin{aligned}P(t_w^*,t_2^*)&=D(S-C)-\frac{D(C_2+\delta(S-C+R))t_2^*}{1+\delta t_2^*}\\&=D(S-C)-DK(t_w^*) \quad (53)\\&>D(S-C)-DK(t_w^{\wedge})=P_1(t_w^{\wedge})\end{aligned}$$

Similarly, we can get $t_2^{\#}>t_2^*$. Then, from equations (33) and (47), we obtain

$$\begin{aligned}P(t_w^*,t_2^*)&=D(S-C)-\frac{D(C_2+\delta(S-C+R))t_2^*}{1+\delta t_2^*}\\&>D(S-C)-\frac{D(C_2+\delta(S-C+R))t_2^{\#}}{1+\delta t_2^{\#}}\end{aligned} \quad (54)$$

Combining equations (53) and (54), we get $P(t_w^*,t_2^*)>\max\left\{P_1(t_w^{\wedge}),P_2(t_2^{\#})\right\}$.

This completes the proof.

Further, let $\pi_1(t_1)$ represent the total profit per unit time in the one-warehouse problem without shortages and let $t_1^{\wedge\wedge}$ denote the optimal solution of $\pi_1(t_1)$; and let $\pi_2(t_2)$ represent the total profit per unit time in the one-warehouse problem without stock and let $t_2^{\#}$ denote the optimal solution of $\pi_2(t_2)$. By using the analogous derivations as in Theorem 4, we can easily obtain the following result. The proof is omitted.

Theorem 5: For the one-warehouse problem, $\pi(t_1^{**},t_2^{**})>\max\{\pi_1(t_w^{\wedge\wedge}),\pi_2(t_w^{\#})\}$

Till now, we present three inventory policies: without shortage, without stock and partial backlogging. From Theorems 4 and 5, we show that the inventory policy with partial backlogging is profitable.

Numerical Results

In order to illustrate the proposed model we provide a computational result for a numerical example with the parameters specified as follows: D=100, A=100, C=10, W=500, C_{11}=0.2, C_{12}=0.5, C_2=2, S=15,α=0.01, β=0.04, θ=0.01, R=7, δ={0.25, 0.5, 1, 2.5,5,∞}

Concluding remarks: In this paper, an inventory model is developed for deteriorating items with finite warehouse capacity, permitting shortage and time-proportional backlogging rate. Holding costs and deterioration costs are different in OW and RW due to different preservation conditions. The inventory costs (including holding cost and deterioration cost) in RW are assumed to be higher than those in OW. To reduce the inventory costs, it will be economical for firms to store goods in OW before RW, but has to empty the stocks in RW before OW. In particular, the backlogging rate considered to be a decreasing function of the waiting time for the next replenishment is more realistic. In practice, we can observe periodically the proportion of demand which would accept backlogging and the corresponding waiting time for the next replenishment. Furthermore, we show that the inventory policy with partial backlogging is more profitable than those without shortage and without stock. We also provide some useful properties for finding the optimal replenishment policy. By using the presented approach, we can easily decide whether the retailer has to rent another warehouse and obtain the optimal replenishment policy among those cases with the help of the auxiliary values. The proposed model can be extended in several ways like we can consider finite rate of replenishment. We can extend the deterministic demand function to stochastic demand patterns. We can also generalize the model to allow for permissible delay in payments.

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Table 1 Effect of δ on the optimal solution

	(δ=0)	δ=0.25	δ=0.5	δ=1	δ=2.5	δ=5	($\delta \to \infty$)
Δ	40.17	46.14	47.65	48.72	49.53	49.84	50.17
t_w^*	0.0619	0.0783	0.0830	0.0866	0.0894	0.0905	0.0916
t_1^*	0.5588	0.5750	0.5797	0.5833	0.5860	0.5871	0.5883
T^*	0.6900	0.6316	0.6158	0.6042	0.5953	0.5919	0.5883
$\frac{t_1^*}{T^*}$	0.8099	0.9104	0.9414	0.9653	0.9844	0.9919	1.0000
Q^*	693.21	634.60	618.96	607.51	598.76	595.43	591.85
B^*	562.02	578.43	583.19	586.78	586.59	590.68	591.85
$P(t_w^*, t_2^*)$	4737.61	4721.1	4716.32	4712.7	4709.87	4708.78	4707.60

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