ON L^1-CONVERGENCE OF FUZZY MODIFIED COSINE SUMS

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Abstract: In this Paper, we generalize Rees and Stanojevic modified cosine sums by introducing a new fuzzy modified cosine sums with fuzzy coefficients and obtain the necessary and sufficient condition for the L^1-convergence of these sums.

Keywords: Fuzzy numbers, fuzzy modified cosine sums, integrability, convergence in fuzzy L^1-norm.

Introduction: Fourier Analysis is a powerful tool for solving many problems and especially for solving various differential equations of interest in science and engineering. The computation and study of Fourier Analysis is known as Harmonic Analysis. It is extremely useful in approximation theory, partial differential equation and probability theory etc. During Literature survey, we found that many authors like C.S Rees and C.V. Stanojevic [2], S. Kumari and B. Ram [20], N. Hooda, S.S. Bhatia and B. Ram [16], K. Kaur, S.S. Bhatia and B. Ram [6], J. Kaur and S.S. Bhatia ([4],[5]), N.L. Braha and Xh. Z. Krasniqi [15] introduced new modified trigonometric sums, as these sums approximate their limits better than the classical trigonometric series in the sense that these sums converge in L^1-metric to the sum of trigonometric series whereas the classical series itself may not.

The development of fuzzy theory gave an idea to extend the classical results in Harmonic Analysis to Fuzzy Analysis. To establish the connection between the Fourier series and Fourier series of fuzzy valued function with the level sets, we studied the basic concepts of fuzzy theory which have been modified and improved by various authors such as L.A. Zadeh [9], O. Kaleva [17], M. Puri and D. Ralescu ([7],[8]), M. Stojakovic ([10],[11],[12]), M. Stojakovic and Z. Stojakovic ([13],[14]), D. Zhang nd C. Guo [3], Ö. Talo and F. Basar [18] and so on. In 2014, U. Kadak and F. Basar [22] have represented the interval valued fuzzy sets in terms of its level sets and also studied the Fourier series of periodic fuzzy valued functions with level sets and examined the convergence of Fourier series of fuzzy valued function. Recently S. Kaur and J. Kaur [21] introduced new classes of fuzzy coefficients and obtained the necessary and sufficient conditions for L^1-convergence of fuzzy trigonometric series.

The aim of this paper to introduce a new fuzzy modified cosine sums with fuzzy coefficients and obtain a necessary and sufficient condition for L^1-convergence of this modified sum.

2. Preliminaries and Background:

In this section, we recall some of the basic notions related to fuzzy numbers.

Definition 2.1. [9] A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $u: R \to [0,1]$ which satisfies the following four conditions:

i. u is normal, i.e., there exists an $x_0 \in R$ such that $u(x_0)=1$.

ii. u is fuzzy convex, i.e., $u[\lambda x + (1-\lambda)y] \ge \min\{u(x),u(y)\}$ for all $x, y \in R$ and for all $\lambda \in [0,1]$.

iii. u is upper semi-continuous.

iv. The set $[u]_0 = \overline{\{x \in R: u(x) > 0\}}$ is compact, where $\overline{\{x \in R: u(x) > 0\}}$ denotes the closure of the set $\{x \in R: u(x) > 0\}$ in the usual topology of R.

we denote the set of all fuzzy numbers on R by E^1 and called it as the space of fuzzy numbers λ-level set $[u]_\lambda$ of $u \in E^1$ is defined by

$$[u]_\lambda = \begin{cases} \{t \in R : u(t) \ge \lambda\} & , 0 < \lambda \le 1 \\ \overline{\{t \in R : u(t) > \lambda\}} & , \lambda = 0 \end{cases}$$

The set $[u]_\lambda$ is closed, bounded and non-empty interval for each $\lambda \in [0,1]$ which is defined by $[u]_\lambda = [u_\lambda^-, u_\lambda^+]$. R can be embedded in E^1, since each $r \in R$ can be regarded as a fuzzy number $\bar{r}$ defined by

$$\bar{r}(x) = \begin{cases} 1 \ , & x = r, \\ 0 \ , & x \neq r. \end{cases}$$

Theorem 2.2. [19] (Goetschel and voxman) For $u \in E^1$, denote $u^-(\lambda) = u^-_\lambda$ and $u^+(\lambda) = u^+_\lambda$. Then

i. $u^-(\lambda)$ is a bounded increasing function on $[0,1]$.
ii. $u^+(\lambda)$ is a bounded decreasing function on $[0,1]$.
iii. $u^-(\lambda) \le u^+(\lambda)$.
iv. $u^-(\lambda)$ and $u^+(\lambda)$ are left continuous on $(0,1]$ and right continuous at 0.
v. If $u^-(\lambda)$ and $u^+(\lambda)$ satisfy (i-iv), then there exist a unique $v \in E^1$ such that $v^-_\lambda = u^-(\lambda)$ and $v^+_\lambda = u^+(\lambda)$.

The above theorem implies that we can identify a fuzzy number u with the parameterized representation

$$\{(u^-_\lambda, u^+_\lambda) \mid 0 \le \lambda \le 1\}$$

Suppose that $u, v \in E^1$ are fuzzy numbers represented by $\{(u^-_\lambda, u^+_\lambda) \mid 0 \le \lambda \le 1\}$ and $\{(v^-_\lambda, v^+_\lambda) \mid 0 \le \lambda \le 1\}$ respectively. If we define

$$(u \oplus v)(z) = \sup_{x+y=z} \min(u(x), v(y)) \qquad (2.1)$$

$$(\alpha u)(z) = \begin{cases} u(z/\alpha), & \alpha \neq 0 \\ \widetilde{0} & \alpha = 0, \ \textit{where} \ \widetilde{0} = \chi_{\{0\}}, \end{cases} \qquad (2.2)$$

then

$$u \oplus v = \{(u^-_\lambda + v^-_\lambda, u^+_\lambda + v^+_\lambda) \mid 0 \le \lambda \le 1\}$$

$$u \Theta v = \left\{ \begin{array}{l} \min(u^-_\lambda - v^-_\lambda, u^+_\lambda - v^+_\lambda), \\ \qquad \max(u^-_\lambda - v^-_\lambda, u^+_\lambda - v^+_\lambda) \mid 0 \le \lambda \le 1 \end{array} \right\}$$

$$\alpha u = \begin{cases} \{(\alpha u^-_\lambda, \alpha u^+_\lambda) \mid 0 \le \lambda \le 1\}, & \alpha \ge 0, \\ \{(\alpha u^+_\lambda, \alpha u^-_\lambda) \mid 0 \le \lambda \le 1\} & \alpha < 0 \end{cases}$$

we define a metric d on E^1 by

$$d(u,v) = \sup_{0 \le \lambda \le 1} d_H([u]_\lambda, [v]_\lambda) \qquad (2.3)$$

where d_H is the hausdorff metric defined as

$$d_H([u]_\lambda, [v]_\lambda) = \max(|u^-_\lambda - v^-_\lambda|, |u^+_\lambda - v^+_\lambda|) \qquad (2.4)$$

Also, $d(u, \widetilde{0})$ will be denoted by $\|u\|$.

Definition 2.3. [18] A sequence $\{u_k\}$ of fuzzy numbers is a function u from the set N into the set E^1. The fuzzy number $\{u_k\}$ denotes the value of the function at $k \in N$ and is called as the general term of the sequence. By $w(F)$, we denote the set of all sequences of fuzzy numbers.

Definition 2.4.[18] A sequence $\{u_n\} \in w(F)$ is called convergent with limit $u \in E^1$, if and only if for every $\in > 0$ there exists an $n_0 = n(\in) \in N$ such that $D(u_n, u) < \in$ for all $n \ge n_0$.

Definition 2.5.[21] A sequence $\{u_n\} \in w(F)$ is said to be decreasing sequence if $u_{k+1} \prec u_k$ i.e. $(u_k)^-_\lambda < (u_{k+1})^-_\lambda$ and $(u_k)^+_\lambda > (u_{k+1})^+_\lambda$

Remark 2.6. If $\{u_k\}$ is a decreasing fuzzy sequence, then it can be easily seen that $(\Delta u_k)^-_\lambda = (u_k)^-_\lambda - (u_{k+1})^-_\lambda = \Delta(u_k)^-_\lambda$ and $(\Delta u_k)^+_\lambda = (u_k)^+_\lambda - (u_{k+1})^+_\lambda = \Delta(u_k)^+_\lambda$.

Definition 2.7.[21] A sequence $\{u_n\} \in w(F)$ is said to be fuzzy null sequence if $\lim_{k \to \infty} u_k = [0]_\lambda$ with respect to the level sets. *i.e.*

$$\left\{ \left(\lim_{k \to \infty} (u_k)^-_\lambda = 0^-_\lambda, \lim_{k \to \infty} (u_k)^+_\lambda = 0^+_\lambda \right) \mid 0 \le \lambda \le 1 \right\} \qquad (2.5)$$

Definition 2.8.[21] A decreasing sequence $\{u_n\} \in w(F)$ is said to belong to class **BV(F)** with respect to the level set if (2.5) is satisfied and the series $\sum_{\oplus\, k=0}^{\infty} |\Delta u_k|$ is convergent.

3. Main Result:

we introduce here a fuzzy modified cosine sums with fuzzy coefficients as

$$g^t_n(x) = \sum_{\oplus\, k=1}^{n} \sum_{\oplus\, j=k}^{n} \Delta u_j \cos kx \qquad (u_0 = 0) \qquad (3.1)$$

where

$$\Delta u_k = u_k \Theta u_{k+1}$$

and obtained L^1- convergence of $g^t_n(x)$ to the fuzzy valued function $f^t(x)$.

Theorem 3.1. *If $\{u_k\}$ be a sequence of fuzzy coefficients belonging to class **BV(F)**, then $g_n^t(x)$ converges to fuzzy valued function $f^t(x)$ in L^1-norm if and only if $\left\{\lim_{n\to\infty} u_n \log n = [0]_\lambda \mid 0 \le \lambda \le 1\right\}$.*

Proof. Consider,

$$g_n^t(x) = \sum_{\oplus\, k=1}^{n} \sum_{\oplus\, j=k}^{n} \Delta u_j \cos kx$$

$$= \left\{ \begin{pmatrix} \sum_{k=1}^{n}\sum_{j=k}^{n} (\Delta u_j)_\lambda^- \cos kx, \\ \sum_{k=1}^{n}\sum_{j=k}^{n} (\Delta u_j)_\lambda^+ \cos kx, \end{pmatrix} \middle| \, 0 \le \lambda \le 1 \right\}$$

Since $\{u_k\}$ is a fuzzy decreasing sequence, Therefore

$$g_n^t(x) = \left\{ \begin{pmatrix} \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta(u_j)_\lambda^- \cos kx, \\ \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta(u_j)_\lambda^+ \cos kx, \end{pmatrix} \middle| \, 0 \le \lambda \le 1 \right\}$$

$$= \{I_1, I_2\}$$

First consider,

$$I_1 = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta(u_j)_\lambda^- \cos kx$$

$$= \sum_{k=1}^{n} [\Delta(u_k)_\lambda^- + \Delta(u_{k+1})_\lambda^- + \ldots + \Delta(u_n)_\lambda^-] \cos kx$$

$$= \sum_{k=1}^{n} [(u_k)_\lambda^- - (u_{n+1})_\lambda^-]$$

$$= \sum_{k=1}^{n} (u_k)_\lambda^- \cos kx - (u_{n+1})_\lambda^- D_n(x)$$

Apply Abel's transformation, we get

$$= \sum_{k=1}^{n-1} \Delta(u_k)_\lambda^- D_k(x) + (u_n)_\lambda^- D_n(x) - (u_{n+1})_\lambda^- D_n(x)$$

$$= \sum_{k=1}^{n-1} (\Delta u_k)_\lambda^- D_k(x) + (u_n)_\lambda^- D_n(x) - (u_{n+1})_\lambda^- D_n(x)$$

Since $D_n(x)$ is bounded in $(0,\pi)$ and by given hypothesis $I_1 < \infty$.

Similarly $I_2 < \infty$.

Hence $\lim_{n\to\infty} g_n^t(x) = f^t(x)$ exists in $(0,\pi)$.

Now, we consider

$$\lim_{n\to\infty} \int_0^\pi | f^t(x) \Theta g_n^t(x) |$$

$$= \lim_{n\to\infty} \sup_{\lambda\in[0,1]} \begin{pmatrix} \int_0^\pi \left| f_\lambda^-(x) - (g_n)_\lambda^-(x) \right| dx, \\ \int_0^\pi \left| f_\lambda^+(x) - (g_n)_\lambda^+(x) \right| dx \end{pmatrix}$$

$$= \lim_{n\to\infty} \sup_{\lambda\in[0,1]} \begin{pmatrix} \int_0^\pi \left| \sum_{k=n+1}^{\infty} (u_k)_\lambda^- \cos kx + (u_{n+1})_\lambda^- D_n(x) \right| dx, \\ \int_0^\pi \left| \sum_{k=n+1}^{\infty} (u_k)_\lambda^+ \cos kx + (u_{n+1})_\lambda^+ D_n(x) \right| dx \end{pmatrix}$$

$$= (J_1, J_2)$$

First consider,

$$J_1 = \lim_{n\to\infty} \sup_{\lambda\in[0,1]} \int_0^\pi \left| \sum_{k=n+1}^{\infty} (u_k)_\lambda^- \cos kx + (u_{n+1})_\lambda^- D_n(x) \right| dx$$

Apply Abel's transformation, we have

$$= \lim_{n\to\infty} \sup_{\lambda\in[0,1]} \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta(u_k)_\lambda^- D_k(x) + (u_{n+1})_\lambda^- D_n(x) - (u_{n+1})_\lambda^- D_n(x) \right| dx$$

$$= \lim_{n\to\infty} \sup_{\lambda\in[0,1]} \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta(u_k)_\lambda^- D_k(x) \right| dx$$

Similarly,

$$J_2 = \lim_{n\to\infty} \sup_{\lambda\in[0,1]} \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta(u_k)_\lambda^+ D_k(x) \right| dx$$

$$\lim_{n\to\infty} \int_0^\pi \left| f^t(x) \Theta g_n^t(x) \right| dx$$

$$= \lim_{n\to\infty} \sup_{\lambda\in[0,1]} \begin{pmatrix} \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta(u_k)_\lambda^- D_k(x) \right| dx, \\ \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta(u_k)_\lambda^+ D_k(x) \right| dx \end{pmatrix}$$

$$\le \lim_{n\to\infty} \sup_{\lambda\in[0,1]} \begin{pmatrix} \left| \sum_{k=n+1}^{\infty} \Delta(u_k)_\lambda^- \right| \int_0^\pi \left| D_k(x) \right| dx, \\ \left| \sum_{k=n+1}^{\infty} \Delta(u_k)_\lambda^+ \right| \int_0^\pi \left| D_k(x) \right| dx \end{pmatrix}$$

Since (see e.g. [1], Vol I, p. 67)

$\int_0^\pi |D_n(x)| dx \approx \ln n$ and

$\left\{ \lim_{n\to\infty} u_n \log n = [0]_\lambda \;\middle|\; 0 \le \lambda \le 1 \right\}.$

Hence the conclusion of main results follows.

Remark 3.2 If $(u_k)^-_\lambda = (u_k)^+_\lambda$ in above theorem then the sequence $\{u_k\}$ of fuzzy numbers becomes the sequence of real numbers.

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