FEKETE-SZEGÖ INEQUALITY FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract: Here we describe some classes of analytic functions and its subclasses by which we will be obtaining sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes and subclasses.

Keywords: Univalent functions, Starlike functions, Close to convex functions and bounded functions.

Introduction : Let $\mathcal{A}$ denote the class of functions of the form

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ (1.1)

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([7], [8]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$, Fekete and Szegö[9] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 3 - 4\mu, if\ \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \leq \mu \leq 1; \\ 4\mu - 3, if \mu \geq 1. \end{cases}$$

(1.2)

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (See Chhichra[1], Babalola[6]).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$$ and satisfying the condition

$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}.$ (1.3)

We denote by $\mathcal{K}$, the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$$ and satisfying the condition

$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}.$ (1.4)

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in \mathbb{E}.$ (1.5)

The class of close to convex functions is denoted by C and was introduced by Kaplan [3] and it was shown by him that all close to convex functions are univalent.

$$S^*(A,B) = \left\{f(z) \in \mathcal{A}; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\} \quad (1.6)$$

$$\mathcal{K}(A,B) = \left\{f(z) \in \mathcal{A}; \frac{(zf'(z))'}{f'(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\} \quad (1.7)$$

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}(A,B)$ is a subclass of $\mathcal{K}$.

We introduce a new subclass as $\left\{\mathbf{f(z)} \in \mathcal{A}; \boldsymbol{\alpha}\frac{\mathbf{zf'(z)}}{\mathbf{f(z)}} + (\mathbf{1}-\boldsymbol{\alpha})\frac{\mathbf{zf(z)}}{\mathbf{2\int_0^z f(z)dz}} \prec \frac{\mathbf{1+Az}}{\mathbf{1+Bz}}; \mathbf{z} \in \mathbb{E}\right\}$ and we will denote this class as $f(z) \in \mathcal{C}(S^*)^{-1}[A,B,\alpha]$.

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form $w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1.$ (1.8)

It is known that $|d_1| \leq 1, |d_2| \leq 1 - |d_1|^2$.(1.9)

Preliminary Lemmas: For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz}\right)$ so that

$$\frac{1+Aw(z)}{1+Bw(z)} = 1 + (A-B)c_1 z + (A-B)(c_2 - Bc_1^2)z^2 + --- \quad (2.1)$$

1. Main Results

Theorem 3.1: Let $f(z) \in \mathcal{C}(S^*)^{-1}[A,B,\alpha]$, then $|a\_3 - \mu a\_2^2| \leq \{(9/(2\alpha+1)^(2)\ [(2(〖A-B)〗^2 (7\alpha+2) - 2B(A-B)(2\alpha+1)^2)/9(3\alpha+1) - \mu(A-B)^2]\quad @;\ if\ \mu \leq (2(A-B)(7\alpha+2) - 2(B+1)(2\alpha+1)^2)/(9(3\alpha+1)(A-B))\quad (3.1)@2(A-B)/((3\alpha+1))\quad ;\quad @\ if\ (2(A-B)(7\alpha+2) - 2(B+1)(2\alpha+1)^2)/(9(3\alpha+1)(A-B)) \leq \mu \leq (2(A-B)(7\alpha+2) - 2(B-1)(2\alpha+1)^2)/(9(3\alpha+1)(A-B))\ (3.2)@(9(A-B)^2)/((2\alpha+1)^2)\ [\mu - (2(A-B)(7\alpha+2) - 2B(2\alpha+1)^2)/9(3\alpha+1)(A-B)]\quad ;\ @if\ \mu \geq (2(A-B)(7\alpha+2) - 2(B-1)(2\alpha+1)^2)/(9(3\alpha+1)(A-B))\qquad (3.3))┤$

The results are sharp.

Proof: By definition of $f(z) \in \mathcal{C}(S^*)^{-1}[A,B,\alpha]$, , we have

$$\alpha\frac{zf'(z)}{f(z)} + (1-\alpha)\frac{zf(z)}{2\int_0^z f(z)dz} = \frac{1+Aw(z)}{1+Bw(z)} \quad ; \quad w(z) \in \mathcal{U}. \qquad (3.4)$$

Expanding the series (3.4), we get

$$\alpha\{1 + a_2 z + (2a_3 - a_2^2)z^2 + - - - \} + (1-\alpha)\left\{1 + \frac{a_2}{3}z + \left(\frac{1}{2}a_3 - \frac{2a_2^2}{9}\right)z^2 + - - -\right\} = \left(1 + (A-B)c_1 z + (A-B)(c_2 - Bc_1^2)z^2 + - - - \right). \qquad (3.5)$$

Identifying terms in (3.5), we get

$$a_2 = \frac{3(A-B)}{2\alpha+1}c_1 \backslash \qquad (3.6)$$

$$a_3 = \frac{2(A-B)}{3\alpha+1}c_2 + 2(A-B)\frac{[(A-B)(7\alpha+2) - B(2\alpha+1)^2]}{(2\alpha+1)^2(3\alpha+1)}{c_1}^2. \qquad (3.7)$$

From (3.6) and (3.7), we obtain

$$a_3 - \mu a_2^2 = \frac{2(A-B)}{3\alpha+1}c_2 + \frac{9(A-B)^2}{(2\alpha+1)^2}\left[\frac{2(A-B)(7\alpha+2) - 2B(2\alpha+1)^2}{9(3\alpha+1)(A-B)} - \mu\right]c_1^2 \qquad (3.8)$$

Taking absolute value, (3.8) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{2(A-B)}{3\alpha+1}|c_2| + \frac{9(A-B)^2}{(2\alpha+1)^2}\left|\frac{2(A-B)(7\alpha+2) - 2B(2\alpha+1)^2}{9(3\alpha+1)(A-B)} - \mu\right||c_1^2|. \qquad (3.9)$$

Using (1.11) in (3.9), we get

$$|a_3 - \mu a_2^2| \leq \frac{2(A-B)}{3\alpha+1}(1 - |c_1|^2) + \frac{9(A-B)^2}{(2\alpha+1)^2}\left|\frac{2(A-B)(7\alpha+2) - 2B(2\alpha+1)^2}{9(3\alpha+1)(A-B)} - \mu\right||c_1^2|$$

$$= \frac{2(A-B)}{3\alpha+1} + \frac{9(A-B)^2}{(2\alpha+1)^2}\left[\left|\frac{2(A-B)(7\alpha+2) - 2B(2\alpha+1)^2}{9(3\alpha+1)(A-B)} - \mu\right| - \frac{2(2\alpha+1)^2}{9(3\alpha+1)(A-B)}\right]|c_1|^2. \qquad (3.10)$$

Case I: $\mu \leq \frac{2(A-B)(7\alpha+2) - 2B(2\alpha+1)^2}{9(3\alpha+1)(A-B)}$. (3.10) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{2(A-B)}{3\alpha+1} + \frac{9(A-B)^2}{(2\alpha+1)^2}\left[\frac{2(A-B)(7\alpha+2) - 2(B+1)(2\alpha+1)^2}{9(3\alpha+1)(A-B)} - \mu\right]|c_1|^2.$$

Subcase I (a): $\mu \leq \frac{2(A-B)(7\alpha+2) - 2(B+1)(2\alpha+1)^2}{9(3\alpha+1)(A-B)}$.

Using (1.11), (3.11) becomes

$$|a_3 - \mu a_2^2| \leq \frac{9}{(2\alpha+1)^2}\left[\frac{2(A-B)^2(7\alpha+2) - 2B(A-B)(2\alpha+1)^2}{9(3\alpha+1)} - \mu(A-B)^2\right]. \qquad (3.12)$$

Subcase I (b): $\mu \geq \frac{2(A-B)(7\alpha+2) - 2(B+1)(2\alpha+1)^2}{9(3\alpha+1)(A-B)}$.

We obtain from (3.11)

$$|a_3 - \mu a_2^2| \leq \frac{2(A-B)}{3\alpha+1}. \qquad (3.13)$$

Case II: $\mu \geq \frac{2(A-B)(7\alpha+2) - 2B(2\alpha+1)^2}{9(3\alpha+1)(A-B)}$

Preceding as in case I, we get

$$|a_3 - \mu a_2^2| \leq \frac{2(A-B)}{3\alpha+1} + \frac{9(A-B)^2}{(2\alpha+1)^2}\left[\mu - \left[\frac{2(A-B)(7\alpha+2) - 2(B-1)(2\alpha+1)^2}{9(3\alpha+1)(A-B)}\right]\right]|c_1|^2. \qquad (3.14)$$

Subcase II (a): $\mu \leq \frac{2(A-B)(7\alpha+2) - 2(B-1)(2\alpha+1)^2}{9(3\alpha+1)(A-B)}$

(3.14) takes the form $|a_3 - \mu a_2^2| \leq \frac{2(A-B)}{3\alpha+1}$ (3.15)

$$|a_3 - \mu a_2^2| \leq \frac{2(A-B)}{(3\alpha+1)};\ if\ \frac{2(A-B)(7\alpha+2) - 2(B+1)(2\alpha+1)^2}{9(3\alpha+1)(A-B)} \leq \mu \leq \frac{2(A-B)(7\alpha+2) - 2(B-1)(2\alpha+1)^2}{9(3\alpha+1)(A-B)} \qquad (3.16)$$

Subcase II (b): $\mu \geq \frac{2(A-B)(7\alpha+2) - 2(B-1)(2\alpha+1)^2}{9(3\alpha+1)(A-B)}$

Preceding as in subcase I (a), we get

$$|a_3 - \mu a_2^2| \leq \frac{9(A-B)^2}{(2\alpha+1)^2}\left[\mu - \frac{2(A-B)(7\alpha+2) - 2B(2\alpha+1)^2}{9(3\alpha+1)(A-B)}\right]. \qquad (3.17)$$

Combining (3.12), (3.16) and (3.17), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by

$$f_1(z) = (1 + az)^b$$

Where $a = \frac{(A-B)(1-\alpha) + 4B(2\alpha+1)^2}{3(2\alpha+1)(3\alpha+1)}$

And $b = \frac{9(A-B)(3\alpha+1)}{(A-B)(1-\alpha) + 4B(2\alpha+1)^2}$

Extremal function for (3.2) is defined by $f_2(z)$, where $f_2(z)$ satisfies

$$\{f_2(z)\}^{\alpha}\left\{\int_0^z f_2(z)dz\right\}^{\frac{1-\alpha}{2}} = \frac{z}{1-z^2}$$

Corollary 3.2: Putting $A = 1, B = -1$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{9}{(2\alpha+1)^2}\left[\frac{8(7\alpha+2)+4(2\alpha+1)^2}{9(3\alpha+1)} - 4\mu\right]; \\ \quad if\ \mu \leq \frac{2(7\alpha+2)}{9(3\alpha+1)(A-B)} \\ \frac{4}{(3\alpha+1)}\ ; \\ if\ \frac{2(7\alpha+2)}{9(3\alpha+1)} \leq \mu \leq \frac{2(7\alpha+2)+2(2\alpha+1)^2}{9(3\alpha+1)} \\ \frac{36}{(2\alpha+1)^2}\left[\mu - \frac{2(7\alpha+2)+(2\alpha+1)^2}{9(3\alpha+1)}\right]\ ; \\ \quad if\ \mu \geq \frac{2(7\alpha+2)+2(2\alpha+1)^2}{9(3\alpha+1)} \end{cases}$$

These are the required results for the class $\mathcal{C}(S^*)^{-1}[\alpha]$

Corollary 3.3: Putting α =1 in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(A-B)^2 - B(A-B)}{2} - \mu(A-B)^2\ ; \\ \quad if\ \mu \leq \frac{(A-B)-(B+1)}{2(A-B)} \\ \frac{(A-B)}{2}\ ; \\ if\ \frac{(A-B)-(B+1)}{2(A-B)} \leq \mu \leq \frac{(A-B)-(B-1)}{2(A-B)} \\ \mu(A-B)^2 - \frac{(A-B)^2 - B(A-B)}{2}\ ; \\ \quad if\ \mu \geq \frac{(A-B)-(B-1)}{2(A-B)} \end{cases}$$

These are the required results for the class $\mathcal{C}(S^*)^{-1}[A,B]$

Corollary 3.4: Putting $A = 1, B = -1\ and\ \alpha$ =0 in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 20 - 36\mu\ ;\ if\ \mu \leq \frac{4}{9} \\ 4\ ;\ if\ \frac{4}{9} \leq \mu \leq \frac{2}{3} \\ 36\mu - 20\ ;\ if\ \mu \geq \frac{2}{3} \end{cases}$$

These estimates were derived by Singh,Saroa and Mehrok [11].

Corollary 3.5: Putting $A = 1, B = -1\ and\ \alpha$ =1 in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if \mu \leq \frac{1}{2}; \\ 1 if \frac{1}{2} \leq \mu \leq 1; \\ 4\mu - 3, if\ \mu \geq 1 \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent starlike functions.

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