FEKETE-SZEGÖ INEQUALITY FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract: In this paper some new classes of analytic functions, its subclasses are introduced. We also obtain sharp upper bounds of the function $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes and subclasses.

Keywords: Univalent functions, Starlike functions, Close to convex functions and bounded functions.

Introduction : Let $\mathcal{A}$ denote the class of functions of the form

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ (1.1)

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([7], [8]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$, Fekete and Szegö[9] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if\ \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \leq \mu \leq 1; \\ 4\mu - 3, if \mu \geq 1. \end{cases} \quad (1.2)$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (See Chichra[1], Babalola[6]).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$$ and satisfying the condition

$$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \quad (1.3)$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$$ and satisfying the condition

$$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \quad (1.4)$$

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \quad (1.5)$$

The class of close to convex functions is denoted by C and was introduced by Kaplan [3] and it was shown by him that all close to convex functions are univalent.

$$S^*(A,B) = \left\{f(z) \in \mathcal{A}; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\} \quad (1.6)$$

$$\mathcal{K}(A,B) = \left\{f(z) \in \mathcal{A}; \frac{(zf'(z))'}{f'(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\} \quad (1.7)$$

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}(A,B)$ is a subclass of $\mathcal{K}$.

In this paper, we establish Fekete-Szegö Inequality for the following subclass of a new class $S^{**}(f,f')$ defined as

$$\left\{f(z) \in \mathcal{A}; \left(\frac{zf'(f(z))f'(z)}{f(f(z))}\right) \prec \frac{1+Az}{1+Bz}; z \in \mathbb{E}\right\}$$

and we will denote this class as $S^*(f,f',A,B)$.

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z)); z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form $w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1$. (1.8)

It is known that

$$|d_1| \leq 1, |d_2| \leq 1 - |d_1|^2. \quad (1.9)$$

Preliminary Lemmas:

For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz}\right)$ so that $\frac{1+w(z)}{1-w(z)} = 1 + 2cz + 2z^2 + \cdots$.

(2.1)

3. Main Results

THEOREM 3.1: Let $f(z) \in S^*(f, f', A, B)$, then

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{A-B}{4}[-B-\mu(A-B)], if \mu \leq \frac{-B-1}{A-B}; & (3.1) \\ \frac{A-B}{4}, if \frac{-B-1}{A-B} \leq \mu \leq \frac{1-B}{A-B}; & (3.2) \\ \frac{A-B}{4}[B+\mu(A-B)], if \mu \geq \frac{1-B}{A-B}; & (3.3) \end{cases}$$

The results are sharp.

PROOF: By definition of $S^*(f, f', A, B)$, we have

$$\left(\left(\frac{zf'(f(z))f'(z)}{f(f(z))}\right) = \frac{1+Aw(z)}{1+Bw(z)}\right); w(z) \in \mathcal{U} \quad (3.4)$$

Expanding the series (3.4), we get

$$(z + 2a_2z^2 + 3a_3z^3 + ----------) + (2a_2z^2 + 4a_2^2z^3 + ----------) + \{(2a_2^2 + 3a_3)z3 + -------\} = \{z + 2a_2z^2 + 2(a_2^2 + a_3)z3 + ------------\} + \{z + (A-B)c_1z^2 + 2a_2(A-B)c_1z^3 + -------\} + \{(A-B)(c_2 - B c_1^2)z^3 + ---------\}$$

(3.5)

Comparing terms in (3.5), we get

$$a_2 = \frac{1}{2}(A-B)c_1 \quad (3.6)$$

$$a_3 = \frac{1}{4}(A-B)[-B\, c_1{}^2 + c_2]$$

(3.7) From (3.6) and (3.7), we obtain

$$a_3 - \mu a_2^2 = \frac{1}{4}(A-B)[c_2 - (B + \mu(A-B)c_1{}^2]$$

(3.8)

Taking absolute value, (3.8) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{4}\{1 + (-B - \mu(A-B) - 1)|c_1^2|\} \quad (3.9)$$

CASE I: $\boldsymbol{\mu} \leq \frac{-B-1}{A-B}$,then (3.9) can be rewritten as

$$\frac{4}{A-B}\left|a_3 - \mu a_2^2\right| \leq 1 + \{(-B-1) - \mu(A-B)\}|c_1^2|\ \} \quad (3.10)$$

SUBCASE I (A): $\boldsymbol{\mu} \leq \frac{-B-1}{A-B}$, using **(1.9)**, (3.10) becomes

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(A-B)}{4}\{-\boldsymbol{B} - \mu(A-B)\}$$

(3.11)

SUBCASE I (B): $\boldsymbol{\mu} \geq \frac{-1-B}{(A-B)}$.

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(A-B)}{4} \quad (3.12)$$

CASE II: $\boldsymbol{\mu} \leq \frac{1-B}{(A-B)}$, then (3.9) can be rewritten as

$$\frac{4}{A-B}\left|a_3 - \mu a_2^2\right| \leq 1 + \{(B-1) + \mu(A-B)\}|c_1^2|\ \} \quad (3.13)$$

SUBCASE II (A): $\boldsymbol{\mu} \leq \frac{1-B}{(A-B)}$ then from equation (3.13) we get $\frac{4}{A-B}\left|a_3 - \mu a_2^2\right| \leq 1$

or $\left|a_3 - \mu a_2^2\right| \leq \frac{A-B}{4}$

(3.14)

SUBCASE II (B): $\boldsymbol{\mu} \geq \frac{1-B}{(A-B)}$ then from equation (3.13) we get

$$\frac{4}{A-B}\left|a_3 - \mu a_2^2\right| \leq \mu(A-B) + B$$

or $\left|a_3 - \mu a_2^2\right| \leq \frac{A-B}{4}\{\mu(A-B) + B\}$

(3.15)

Combining subcase I(b) and subcase II(a), we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \frac{A-B}{4}\ if\ \frac{-B-1}{A-B} \leq \boldsymbol{\mu} \leq \frac{1-B}{(A-B)}$$

(3.16)

Combining (3.12), (3.14) , (3.15) and (3.16), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by $f_1(z)$ where

$$\boldsymbol{f_1\{f_1(z)\} = z(1 + Bz)^{\frac{B-A}{B}}}$$

Extremal function for (3.2) is defined by $f_2(z)$ where

$$\boldsymbol{f_2\{f_2(z)\} = z(1 + Bz^2)^{\frac{B-A}{B}}}$$

Corollary 3.2: Putting $A = 1, B = -1$ in the theorem, we get

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{1}{2}[1-2\mu], if \mu \leq 0; \\ \frac{1}{2}, if\ 0 \leq \mu \leq 1; \\ \frac{1}{2}[2\mu - 1], if \mu \geq 1; \end{cases}$$

These are the results for the class $S^*(f, f')$.

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