CONSISTENCY OF PAIR-WISE COMPARISION MATRIX IN ANALYIC HIERARCHY PROCESS – AN ILLUSTRATION

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Abstract: Consistency is a major part in Analytic Hierarchy Process. AHP allows some small inconsistency (i.e.10%) in judgments. If a pair-wise comparison matrix fails to satisfy the consistency condition, decision maker needs to review the judgments. There are some methods to improve the Consistency Ratio of the pair-wise comparison matrix; these are too difficult to apply. This paper proposes an easy method to improve the Consistency Ratio of a pair-wise comparison matrix. Matrix multiplication, vector dot product and definitions of consistent pair-wise comparison matrix are used to identify the inconsistent elements.

Key Words: AHP, Consistency, Multi Criteria Decision Making, Pair-wise Comparison,

Introduction: The Analytic Hierarchy Process was introduced by Thomas.L.Saaty in 1970s. It is one of the most widely used decision making methods, which has been applied to various decision making situations. The pair-wise comparison matrix which is obtained by the decision maker should be consistent, however some small inconsistency (i.e.10%) may be allowed. If the inconsistency is more than 10% it will impact on the result of priority vector. This paper explains:

(i). An easy method to identify the inconsistent elements of the pair-wise comparison matrix

(ii). Inconsistent elements can be identified accurately while more original information can be preserved

(iii). This Proposed method can be applied to identify the inconsistent elements for higher order comparison matrices.

Methodology:

The basic procedure to carry out the AHP consist the following steps:

(a). Structuring a decision problem and selection of criteria

(b). Priority setting of the criteria by pair- wise comparison (weighing)

(c). Pair -wise comparison of options on each criterion (scoring)

(d). Obtaining an overall relative score for each option

2.1. Definitions and notations for the pair-wise comparison matrix:

Definition 1 .A comparison matrix A is said to be positive reciprocal if

$a_{ij} = 1, a_{ij} > 0 \; and \; a_{ij} = 1/a_{ji}$

Definition 2. A positive reciprocal matrix is perfectly consistent if $a_{ik} \times a_{kj} = a_{ij}$ for all i, j and k

Definition 3. A positive reciprocal matrix is approximately consistent if $a_{ik} \times a_{kj} \cong a_{ij}$ for all i, j and k, where '$\cong$' denotes approximately or close to.

Definition 4. A positive reciprocal matrix is said to be transitive if A>B and B>C then A>C.

Definition 5. The pair-wise comparison matrix can pass the consistency test, if the consistency ratio C.R = $\frac{C.I}{R.I}$ < 0.1, where the consistency index (C.I) = $\frac{\lambda_{\max} - 1}{n - 1}$, R.I is the average random index based on matrix size, $\lambda_{\max}$ is the maximum Eigen value of matrix A, and n is the order of matrix A (Saaty,1991).

Table1: Saaty's Ratio scale for pair wise comparison of importance of weights of Criteria/alternatives

Intensity of importance	Definition	Explanation
1	Equal importance	Two elements contribute equally to the property
3	Moderate importance of one over another	Experience and judgment slightly favor one over the other
5	Essential or strong importance	Experience and judgment strongly favor one over another
7	Very strong importance	An element is strongly favored and its dominance is demonstrated in practice.
9	Extreme importance	The evidence favoring one element over another is one of the highest possible order of affirmation
2,4,6,8	Intermediate values between two adjacent judgments	Comprise is needed between two judgments
Reciprocals	When activity i compared to j is assigned one of the above numbers, the activity j compared to i is assigned its reciprocal	
Rational	Ratios arising from forcing consistency of judgments	

Table 2: Average Random Index (R.I) based on matrix size (adopted from Saaty,2000)

Size of the matrix	Random consistency index(R.I)
1	0
2	0
3	0.52
4	0.89
5	1.11
6	1.25
7	1.35
8	1.40
9	1.45
10	1.49

1.2. The theorems of the inconsistency identification method:

According to the theorems of matrix multiplication and vectors dot product as well as the definitions for reciprocal matrix, we have the following statements.

Theorem 1. The induced matrix C = AA – nA should be a zero matrix if comparison matrix A is perfectly consistent.

Corollary 1. The induced matrix C = AA – nA should be as close as possible to zero matrix if comparison matrix A is approximately consistent.

Corollary 2. There must be some inconsistent elements in induced matrix C deviating far away from zero if the pair – wise matrix is inconsistent.

2.3 The inconsistency identification process:

Assuming the pair – wise comparison matrix A with n rows and n columns is inconsistent. Based on the above definitions, theorems and corollaries, the processes to identify inconsistent elements of comparison matrix as well as the methods to analyze and adjust those elements are proposed as the following three major steps which include seven specific identifying steps.

Step 1: Identify the location of inconsistent element whose absolute value is the largest in the induced pair – wise comparison matrix.

Step I: Construct an induced matrix C with the following formula.

$C = AA - nA.$

Step 2: Identify the largest absolute value (s) of elements deviating farthest from zero in the induced matrix C, and record the location. For instance, suppose C_{ij} is such an element in matrix C and the location is i^{th} row and j^{th} column.

Step II : Identify the potential inconsistent elements by the bias identifying vector.

Step 3: Let the i^{th} row of the original pair – wise comparison matrix A be represented as row vector $r_i = (a_{i1}.a_{i2}.......a_{in})$ *and the* j^{th} column of the same matrix as a column vector $C_j^T = (a_{1j}, a_{2j} ....a_{nj})^T$ Where C_j^T is the transpose vector of column vector C_j.

Step 4: Calculate the scalar product of the vectors r_1 and C_j^T in n dimension. The dot product b of the two vectors becomes:

$$b = r_i.C_j^T = (a_{i1}.a_{i2}.....a_{in}).(a_{1j}, a_{2j}.....a_{nj}) = (a_{i1}a_{1j}, a_{i2}a_{2j},...a_{in}a_{nj})$$

Step 5: Compute the deviation elements which are far away from a_{ij} in vector b by the following formula. Let f be the bias identifying vector henceforth.

$$f = b - a_{ij} = (a_{i1}a_{1j} - a_{ij}, a_{i2}a_{2j} - a_{ij}, ...a_{in}a_{nj} - a_{ij})$$

Step III : Identify the inconsistent elements using the identification method and the method of matrix order reduction.

Step 6: Identify the error elements in pair-wise matrix A that might cause the inconsistency by bias identifying vector f using the following three principal identification methods and the method of matrix order reduction.

(a). Method of maximum: If more absolute values in vector f are around zero, and fewer values are deviating from zero, then identify the largest value in vector f. If there are other values close to the largest one, then identify those elements simultaneously.

(b). Method of minimum: If more absolute values of elements in vector f are far away from zero, and fever values are close to zero, or equal to zero, then identify the smallest value in vector f. If there are other values close or equal to the smallest one, then identify those elements simultaneously.

(c). Method of identifying a_{ij} : (1). If the largest value in the induced matrix C is negative, then a_{ij} is too large. (2). If there are only two zeroes where the location is i[th] and j[th] in bias vector f, and others are positive, then a_{ij} is too small. Otherwise a_{ij} is too large. In the former case, if a_{ij} is already close to the maximum scale9, then identify the next largest value in the induced matrix C, and further identify other inconsistent elements using method of matrix order reduction.

Assume the bias value of $a_{ik}a_{kj} - a_{ij}$ in bias vector f is the largest positive one, and others are around zero. Clearly, $a_{ik}a_{kj}$ is larger than a_{ij}, and others are equal or close to a_{ij}. Then, there are following four conditions.

Condition 1: a_{ik} is too large;

Condition 2: a_{kj} is too large;

Condition 3: both a_{ik} and a_{kj} are too large; or

Condition 4: a_{ij} is too small.

Given the above conditions, the element to be adjusted could be identified as step 7:

D.Ergu et al./ European Journal of Operations Research 213(2011)246-259

Table 3: The identification process for method of matrix order reduction (sub-step 1-3)

Remove	**Test**	**Might problem**	**Good**	**Problem**
A_k	√	a_{ik}, a_{kj}	a_{ij}	
	×	a_{ik}, a_{kj}		a_{ij}
A_i	√	a_{ik}	a_{kj}	
	×			a_{kj}
A_j	√	a_{kj}	a_{ik}	
	×			a_{ik}

Step 7: Find the values of C_{ik} and C_{kj} in the induced matrix C according to the following procedure:

Based on our assumption, that is, assume the bias value of $a_{ik}a_{kj} - a_{ij}$ in bias vector f is the largest positive one, then one or both a_{ik} and a_{kj} are too large, therefore it is impossible that $C_{ik} > 0$ and $C_{kj} > 0$ simultaneously.

If $C_{ik} < 0$ and $C_{kj} > 0$, then a_{ik} is too large due to $C_{ik} = \frac{1}{n}\sum_{l=1}^{n} a_{il}a_{lk} - a_{ik}$, and a_{kj} be too small.

If a_{jk} is too large, then the decision makers should decrease the value of element a_{ik}. So the value of $a_{ik}a_{kj}$ is closer to the value of a_{ij}.

If $C_{ik} < 0\, and\, C_{kj} < 0,$ and the bias between both absolute values are too large, then the maximum absolute element can be identified using the inconsistency identification method again.

If $C_{ik} < 0\, and\, C_{kj} < 0,$ and the bias between both absolute values are close to each other, then the following method of matrix order reduction for pair – wise matrix could be used to identify the bias elements. This method could identify the bias elements accurately and keep the comparison information provided by the experts as much as possible, especially for the pair – wise matrix with high order. The method of matrix order reduction could also identify the elements which are close to the largest or smallest simultaneously.

2.3.1. Method of matrix order reduction.

As illustrated above, both a_{ik} and a_{kj} are either too large or the value of a_{ij} is too small. It indicates that some attributes or criteria, namely, $A_i, A_k\, or\, A_j$ have impacts on other attributes and is an inconsistent element. Therefore, we can test whether it can pass the consistency test or not by removing some attributes one by one from the original pair – wise matrix, which is called method of matrix order reduction. The inconsistent attributes could be identified by this method with the following sub – steps:

Sub-step1: Test the consistency of the order reduced comparison matrix $A_{(n-1)\times(n-1)}$ by removing the attribute A_k, namely, deleting k^{th} row and k^{th} column from the original pair – wise matrix A.

If the consistency test passed, the attribute A_k is inconsistent while a_{ij} is consistent, then go to Sub – step 2 to identify a_{ik} and a_{kj}.

If the consistency test failed, there must be other inconsistent attributes in the order reduced pair– wise matrix. Hence, a_{ij} is inconsistent and the value of a_{ij} can be increased so it is closer to the average of $\sum_{k=1}^{n} a_{ik}.a_{kj}$. Meanwhile, both a_{ik} and a_{kj} also might be problematic, and continue to sub – step 2

Sub-step2: Test the consistency of the order reduced pair – wise matrix $A_{(n-1)\times(n-1)}$ by removing the attribute A_i from the original pair – wise matrix A.

If the consistency test passed, both attributes A_k and A_i are consistent. There is no need to change a_{kj}. Hence, decrease a_{ik} as a_{ij} was identified in sub – step 1.

If the consistency test failed, at least one of the attributes A_k or A_j is inconsistent, then decrease a_{kj}. Meanwhile, a_{ik} might also be inconsistent and go to sub – step 3.

Sub-step3: Test the consistency of order reduced pair – wise matrix $A_{(n-1)\times(n-1)}$ by removing the attribute A_j from the original pair – wise matrix A.

If the consistency test passed, then a_{ik} is consistent; otherwise a_{ik} should be decreased.

If the consistency test failed in both sub – steps 2 and 3. We have to let the decision makers to change both elements a_{ik} and a_{kj} simultaneously.

If the decision makers want to further check whether there exists other inconsistent attributes. We have to test the consistency of the order reduced pair – wise matrix by removing attributes $A_i, A_k\, or\, A_j$ simultaneously.

To explain the identification process, Table3 shows the identification process of $a_{ik}a_{kj}$ or a_{ij} in Table3, "Remove" represents removing the corresponding attributes. "Test" denotes the consistency test for the order reduced pair – wise matrix. "Might Problem" stands for the elements might have inconsistent problem. "Good" denotes the elements are consistent. "Problem" denotes the elements are inconsistent. "×" denotes the consistency test failed while "√" stands for a passed consistency test. $A_i, A_k\, and\, A_j$ Stand for three different attributes.

2. Illustrative Example:

In order to test and compare the consistency identification method with other methods, we applied the proposed method to improve the C.R of the following pair-wise comparison matrix, which is taken from mathematica aeterna, volume.2, 2012, no.861-878. ('Analytic Hierarchy Process approach-An application of engineering education')

Example: The 5 × 5 pair-wise comparison matrix A is inconsistent with C.R = 0.2041 > 0.1.

$$A = \begin{pmatrix} 1 & 1 & 3 & 1 & .33 \\ 1 & 1 & 3 & 3 & 5 \\ .33 & .33 & 1 & .33 & 1 \\ 1 & .33 & 3 & 1 & 3 \\ 3 & .2 & 1 & .33 & 1 \end{pmatrix}$$

The proposed method is applied to test this pair-wise comparison matrix following (Steps 1-7) in section 2.3 were done by MATLAB.

Step 1: The induced matrix C = A*A – 5*A is

$$C = \begin{pmatrix} -0.02 & -1.61 & -2.67 & -1.10 & 10.01 \\ 15.99 & -0.02 & 8 & -5.36 & -2.67 \\ 2.67 & -0.35 & -0.03 & 0.66 & -0.25 \\ 7.32 & 1.6 & -2.01 & -0.03 & -4.02 \\ -8.14 & 2.84 & 7.59 & 2.94 & -0.02 \end{pmatrix}$$

Step 2: The largest value in C is 15.99, where location is 2nd row and 1st column.

Step 3: Draw out all the values in 2nd row and 1st column of pair-wise matrix A, that is

$r_2 = (1 \quad 1 \quad 3 \quad 3 \quad 5)\, and\; c_1^T = (1 \quad 1 \quad .33 \quad 1 \quad 3)$

Step 4: The scalar product b of the vectors $r_2\, and\; c_1^T$ in the dimension 5, that is

b = $r_2 . c_1^T = (1 \quad 1 \quad .99 \quad 3 \quad 15)$

Step 5: The bias identifying vector *f* is *f* = $b - a_{21} = (0 \quad 0 \quad -.01 \quad 2 \quad 14)$

Step 6 : The value 14, is the largest far from zero. It indicates that a_{21} = 1 is probably correct while 14 = $a_{25} \times a_{51} - a_{55}$ is the inconsistent element.

Therefore, we identified $a_{25}\, a_{51}$ may have problem.

Step 7: As $c_{25} = -2.67 < 0\, and\; c_{51} = -8.14 < 0$ and the corresponding elements $a_{25}\, and\; a_{51}$ are too large. Then, the method of matrix order reduction is applied to identify $a_{25}\, and\; a_{51}$

Sub- step 1: Remove 2nd row and 2nd column from pair-wise comparison matrix A, and do the consistency test, the

$\lambda_{max} = 4.6501\, and\; CR = 0.24 > 0.1$, the test failed.

Check a_{51} and decrease the value of a_{51} and let the product of

$a_{25} \times a_{51}$ *as close to* $a_{21} = 1$ *as possible* .

Sub- step 2: Remove 1st row and 1st column from pair-wise comparison matrix A, and do the consistency test, the

$\lambda_{max} = 4.1060\, and\; CR = 0.04 < 0.1$, the test passed. So no further correction is needed for a_{25} .

Sub- step 3: Remove 5th row and 5th column from pair-wise comparison matrix A, and do the consistency test, the

$\lambda_{max} = 4.1453\, and\; CR = 0.05 < 0.1$, the test passed. So no further correction is needed for a_{21}

Most of the time, we do not need to finish all Sub-steps for inconsistency test, except some situations when complicated inconsistency identification and adjustment is needed.

3. Conclusions And Implications:

In this paper we have proposed a simple method to identify the inconsistent elements in the pair-wise comparison matrix. Matrix multiplication and vectors dot product as well as the definitions of the pair-wise comparison matrix were used. The effectiveness of the method has also been explained with one example which is taken from mathematica aeterna, volume.2, 2012, no.861-878. ('Analytic Hierarchy Process approach-An application of engineering education')The inconsistency identification process can be conducted in three important steps. The inconsistent elements could be easily identified in the first step. Number of possible inconsistent elements could be identified directly in the second step. Then the specific inconsistent elements could be identified and revised in the third step.

We can apply this proposed method to identify inconsistent elements for higher order comparison matrices till 10th order.

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