COMMON FIXED POINT THEOREMS IN G-METRIC SPACES USING THE CONCEPT OF COMPATIBLE CONTINUITY

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Abstract: In this paper, we use the settings of generalized metric spaces to obtain common fixed point results for a pair of maps using the concept of compatible continuity, weakly compatible maps.

Keywords: Metric space, G-metric space, compatible maps, weakly compatible maps, fixed point.

Introduction: The fixed point theorems in metric spaces are playing a major role to construct methods in mathematics to solve problems in applied mathematics and sciences. So the attraction of metric spaces to a large numbers of mathematicians is understandable. Some generalizations of the notion of a metric space have been proposed by some authors.

Jungck [1] proved a common fixed point theorem for commuting maps, which generalized the Banach fixed point theorem. This theorem has many applications but suffers from one drawback that the continuity of a map throughout the space is needed. Jungck [2] defined the concept of compatible mappings.

In 1992, Dhage [3] introduced the concept of a D-metric space. Geometrically, a D-metric D(x, y, z) represents the perimeter of the triangle with vertices x, y, and z in R^2. Recently, Mustafa and Sims [4] have shown that most of the results concerning Dhage's D-metric spaces are invalid. Therefore, they introduced an improved version of the generalized metric space structure, which they called G-metric spaces.

In 2006, Mustafa and Sims [5] introduced the concept of G-metric spaces as follows.

Definition 1.1[5] Let X be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric or , more specifically, a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 1.2[5] Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points in X. A point x in X is said to be the limit of the sequence $\{x_n\}$, $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$, and one says that the sequence $\{x_n\}$ is G-convergent to x.

Thus, $x_n \to x, n \to \infty$ or $\lim_{n\to\infty} x_n = x$ in a G-metric space (X, G) if for each $\varepsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \varepsilon$ for all $m, n \geq N$.

Proposition 1.1 [5] Let (X, G) be a G-metric space then the following are equivalent:

1. $\{x_n\}$ is G-convergent to x,
2. $G(x_n, x_n x) \to 0$ as $n \to \infty$,
3. $G(x_n, x, x) \to 0$ as $n \to \infty$,
4. $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Definition 1.3 [5] Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if, for each $\varepsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$. i.e., if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 1.2 [6] If (X, G) is a G-metric space, then the following are equivalent:

1. the sequence $\{x_n\}$ is G-Cauchy,
2. For each $\varepsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \varepsilon$ for all $m, n \geq N$.

Proposition 1.3 [5] Let (X, G) be a G-metric space then the function $G(x, y, z)$ is jointly continuous in all three variables.

Definition 1.4 [5] A G-metric space (X, G) is called a symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all x, y in X.

Proposition 1.4 [7] Every G-metric space (X, G) will define a metric space (X, d_G) by

1. $d_G(x, y) = G(x, y, y) + G(y, x, x)$ for all x, y in X.

If (X, G) is asymmetric G-metric space, then

2. $d_G(x, y) = 2G(x, y, y)$ for all x, y in X.

However, if (X, G) is not symmetric, then it follows from the G-metric properties that

3. $\frac{3}{2} G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y)$ for all x, y in X.

Definition 1.5 [7] A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in X.

Proposition 1.5 [7] A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, d_G) is complete metric space.

Proposition 1.6 [6] Let (X, G) be a G-metric space. Then, for any $x, y, z, a \in X$, it follows that

1. If $G(x, y, z) = 0$ then $x = y = z$,
2. $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
3. $G(x, y, y) \leq 2G(y, x, x)$,
4. $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
5. $G(x, y, z) \leq \frac{2}{3}\big(G(x, y, a) + G(x, a, z) + G(a, y, z)\big)$,
6. $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$.

Definition 1.6 [8] A pair of self-mappings (f, g) of a G- metric space (X, G) is said to be compatible if $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$ or $\lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$ for some z in X.

Definition 1.7 [9] A pair of self-mapping (f, g) of a G-metric space (X, G) is said to be R-weakly commuting at a point x in X if $G(fgx, gfx, gfx) \leq RG(fx, gx, gx)$ for some R > 0.

Definition 1.8 [9] Two self-maps f and g of a G-metric space (X, G) are called pointwise R-weakly commuting on X if, given x in X, there exists R > 0 such that $G(fgx, gfx, gfx) \leq RG(fx, gx, gx)$.

Main results

Theorem 2.1 Let (X, G) be a complete G- metric space and f, g be two self-mappings on (X, G) satisfies the following conditions;

1.$f(X) \subseteq g(X)$(1)

2.$gorf$ is continuous(2)

3.
$$G(fx, fy, fz) \leq aG(gx, gy, gz) + bG(fx, gx, gx) + cG(fy, gy, gy) + dG(fz, gz, gz) + e\max\{G(fx, gy, gy), G(fy, gx, gx), G(fy, gz, gz), G(fz, gy, gy), G(fz, gx, gx), G(fx, gz, gz)\} \quad (3)$$

For every $x, y, z \in X$ and $a, b, c, d, e > 0$with $0 \leq a + 2b + 2c + 2d + 4e \leq 1$. Then f and g have a unique common fixed point in X provided f and g are compatible maps.

Proof: Let $x_0 \in X$. Since $f(x) \subseteq g(x)$, there exists a sequence of points $\{x_n\}$ such that $fx_n = gx_{n+1}$.

Define a sequence $\{y_n\}$ in X by

$$y_n = fx_n = gx_{n+1}.$$

Now we will show that $\{y_n\}$ is a G-Cauchy sequence in X. For proving this, by (3) take $x = x_n$, $y = x_{n+1}$, $z = x_{n+1}$, we have

$$G(fx_n, fx_{n+1}, fx_{n+1}) \leq aG(gx_n, gx_{n+1}, gx_{n+1}) + bG(fx_n, gx_n, gx_n) + cG(fx_{n+1}, gx_{n+1}, gx_{n+1}) + dG(fx_{n+1}, gx_{n+1}, gx_{n+1}) + e\max\{G(fx_n, gx_{n+1}, gx_{n+1}), G(fx_{n+1}, gx_n, gx_n), G(fx_{n+1}, gx_{n+1}, gx_{n+1}), G(fx_{n+1}, gx_{n+1}, gx_{n+1}), G(fx_{n+1}, gx_n, gx_n), G(fx_n, gx_{n+1}, gx_{n+1})\}$$

$$G(y_n, y_{n+1}, y_{n+1}) \leq aG(y_{n-1}, y_n, y_n) + bG(y_n, y_{n-1}, y_{n-1}) + cG(y_{n+1}, y_n, y_n) + dG(y_{n+1}, y_n, y_n) + e\max\{G(y_n, y_n, y_n), G(y_{n+1}, y_{n-1}, y_{n-1}), G(y_{n+1}, y_n, y_n), G(y_{n+1}, y_n, y_n), G(y_{n+1}, y_{n-1}, y_{n-1}), G(y_n, y_n, y_n)\}$$

$$G(y_n, y_{n+1}, y_{n+1}) \leq aG(y_{n-1}, y_n, y_n) + 2bG(y_{n-1}, y_n, y_n) + (c + d)G(y_{n+1}, y_n, y_n) + e\max\{G(y_{n+1}, y_{n-1}, y_{n-1}), G(y_{n+1}, y_n, y_n)\}$$

{By proposition 1.6}

$$G(y_n, y_{n+1}, y_{n+1}) \leq aG(y_{n-1}, y_n, y_n) + 2bG(y_{n-1}, y_n, y_n) + (c + d)2G(y_n, y_{n+1}, y_{n+1}) + e\max\{G(y_{n+1}, y_{n-1}, y_{n-1}), G(y_{n+1}, y_n, y_n)\} \quad (4)$$

{By proposition 1.6}

But by (G5), we have

$$G(y_{n+1}, y_{n-1}, y_{n-1}) \leq G(y_{n+1}, y_n, y_n) + G(y_n, y_{n-1}, y_{n-1})$$

$$G(y_{n+1}, y_{n-1}, y_{n-1}) \leq 2G(y_n, y_{n+1}, y_{n+1}) + 2G(y_{n-1}, y_n, y_n)$$

{By proposition 1.6}

So, (4) becomes

$$G(y_n, y_{n+1}, y_{n+1}) \leq (a + 2b)G(y_{n-1}, y_n, y_n) + (c + d)2G(y_n, y_{n+1}, y_{n+1}) + e\,2\{G(y_n, y_{n+1}, y_{n+1}) + G(y_{n-1}, y_n, y_n)\}$$

$$(1-2c-2d-2e)G(y_n, y_{n+1}, y_{n+1}) \leq (a+2b+2e)G(y_{n-1}, y_n, y_n)$$

$$G(y_n, y_{n+1}, y_{n+1}) \leq \frac{a+2b+2e}{1-2c-2d-2e} G(y_{n-1}, y_n, y_n)$$

That is, $G(y_n, y_{n+1}, y_{n+1}) \leq G(y_{n-1}, y_n, y_n)$, where $q = \frac{\alpha+\delta+\mu}{1-2\beta-2\rho-2\delta-2\mu} < 1$

Continuing in the same way, we have

$$G(y_n, y_{n+1}, y_{n+1}) \leq q^n G(y_{n-1}, y_n, y_n)$$

Therefore, for all $n, m \in N, n < m$, we have by rectangle inequality that

$$\begin{aligned} G(y_n, y_m, y_m) &\leq G(y_n, y_{n+1}, y_{n+1}) \\ &+ G(y_{n+1}, y_{n+2}, y_{n+2}) \\ &+ \cdots \ldots \ldots + G(y_{m-1}, y_m, y_m) \\ &\leq (q^n + q^{n+1} + \cdots \ldots + q^{m-1}) G(y_0, y_1, y_1) \\ &\leq \frac{q^n}{1-q} G(y_0, y_1, y_1) \end{aligned}$$

Letting as $n, m \to \infty$, we have $\lim_{n\to\infty} G(y_n, y_m, y_m) = 0$. Thus $\{y_n\}$ is G-Cauchy sequence in X. Since (X, G) complete G-metric space, therefore, there exists a point $z \in X$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z$ since the mapping f or g is continuous, for definiteness one can assume that g is continuous, therefore, $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gz$. Further, f and g are compatible,

therefore, $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0 \Rightarrow \lim_{n\to\infty} fgx_n = gz$.

From (3), we have

$$\begin{aligned} G(fgx_n, fx_n, fx_n) &\leq aG(ggx_n, gx_n, gx_n) \\ &+ bG(fgx_n, ggx_n, ggx_n) \\ &+ cG(fx_n, gx_n, gx_n) \\ &+ dG(fx_n, gx_n, gx_n) \end{aligned}$$

+□ max{G(□□□□, □□□, □□□),
□(□□□, □□□□, □□□□),
□(□□□, □□□, □□□),
□(□□□, □□□, □□□),
□(□□□, □□□□, □□□□),
□(□□□□, □□□, □□□)}

Taking limit as □ → ∞ we have, □□ = □.

Again from (3), we have

□(□□□, □□, □□) ≤ □□(□□□, □□, □□)
+□□(□□□, □□□, □□□)
+□□(□□, □□, □□)

$$+dG(fz, gz, gz)$$
$$+e \max\{G(fx_n, gz, gz), G(fz, gx_n, gx_n), G(fz, gz, gz), G(fz, gz, gz), G(fz, gx_n, gx_n), G(fx_n, gz, gz)\}$$

Taking limit $n \to \infty$, we have $z = fz$. Therefore, we have $gz = fz = z$. Thus z is common fixed point of f and g.

For uniqueness, we assume that $z (\neq z_1)$ be another fixed point of f and g. Then $G(z, z_1, z_1) > 0$ and

$$\begin{aligned} G(z, z_1, z_1) = G(fz, fz_1, fz_1) &\leq aG(gz, gz_1, gz_1) \\ &+ bG(fz, gz, gz) \\ &+ cG(fz_1, gz_1, gz_1) \\ &+ dG(fz_1, gz_1, gz_1) \\ &+ e \max\{G(fz, gz_1, gz_1), \\ &G(fz_1, gz, gz), \\ &G(fz_1, gz_1, gz_1), \\ &G(fz_1, gz_1, gz_1), \\ &G(fz_1, gz, gz), \\ &G(fz, gz_1, gz_1)\} \end{aligned}$$

$\leq (a+2e)G(z, z_1, z_1)$, is a contradiction,

Which demands that $z = z_1$.

This is complete proof of the theorem.

Theorem 2.2 Let f and g be two weakly compatible self-maps of a G-metric space (X, G) satisfying conditions (1) and (3) and any one of the subspace $f(X)$ or $g(X)$ is complete. Then f and g have unique common fixed point in X.

Proof. From theorem 2.1, we conclude that $\{y_n\}$ is a G-Cauchy sequence in X. Since $f(X)$ or $g(X)$ is complete, for definiteness assume that $g(X)$ is complete subspace of X then the subsequence of $\{y_n\}$ must get a limit in $g(X)$. Call it be z. Let $u \in g^{-1}z$. Then $gu = z$ as $\{y_n\}$ is a G-Cauchy sequence containing a convergent subsequence, therefore the sequence $\{y_n\}$ also convergent imply thereby the convergence of subsequence of the convergent sequence. Now we show that $fu = z$,

On setting $x = u, y = x_n$ and $z = x_n$, in (3), we have

$$\begin{aligned} G(fu, fx_n, fx_n) &\leq aG(gu, gx_n, gx_n) \\ &+ bG(fu, gu, gu) \\ &+ cG(fx_n, gx_n, gx_n) \\ &+ dG(fx_n, gx_n, gx_n) \\ &+ e \max\{G(fu, gx_n, gx_n), \\ &G(fx_n, gu, gu), \\ &G(fx_n, gx_n, gx_n), \\ &G(fx_n, gx_n, gx_n), \\ &G(fx_n, gu, gu), \\ &G(fu, gx_n, gx_n)\} \end{aligned}$$

Letting as $n \to \infty$ in the above in equality, we have

$$G(fu, z, z) \leq (b+c+d+e)G(fu, z, z)$$

Which implies that, $fu = z$.

Therefore, $fu = gu = z$, i.e., u is a common fixed point of f and g. Since f and g are weakly compatible, it follows that $fgu = gfu$, i.e. $fz = gz$.

We now show that $fz = z$. Suppose that $fz \neq z$, therefore $G(fz, z, z) > 0$. From (3) on setting $x = z, y = u, z = u$, we have

$$
\begin{aligned}
G(fz,z,z) = G(fz,fu,fu) \leq\ & aG(gz,gu,gu) \\
&+bG(fz,gz,gz) \\
&+cG(fu,gu,gu) \\
&+dG(fu,gu,gu) \\
&+e\max\{G(fz,gu,gu), \\
&G(fu,gz,gz), \\
&G(fu,gu,gu), \\
&G(fu,gu,gu), \\
&G(fu,gz,gz), \\
&G(fz,gu,gu)\}
\end{aligned}
$$

$$G(fz,z,z) \leq (b+c+d+e)G(fz,z,z)$$

Which implies that $fz = z$.

Therefore, $fz = gz = z$ i.e., z is common fixed point of $f$and g. Uniqueness follows easily.

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