ON GENERALIZED HERMITE POLYNOMIALS OF THREE VARIBALES

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Abstract: In this paper, we address the problem of framing three variable generalized Hermite polynomials (G.H.P.), H_n (x,y,z) and using some basic relation we obtain linear generating function for three variable generalized Hermite polynomials (G.H.P.).

Key words: Dattoli polynomials, Generalized Hermite polynomials (G.H.P.), Generating functions, Hermite Polynomials, Linear Generating functions,.

Introduction: We consider the three variable generalized Hermite polynomials (G.H.P.) $H_n(x,y,z)$ defined by the generating function [2; p.511(17)] and also by Pathan, Khan and Yasmin [7]

$$\sum_{n=0}^{\infty} H_n(x,y,z)\frac{t^n}{n!}=\exp(2xt-yt^2+zt^3) \quad ...(1.1)$$

These polynomials (1.1) can be viewed as a generalization of two variables Hermite polynomials (G.H.P.), $H_n(x,y)$, defined by the generating function for z = 0 [2, p.510 (8)] (and see also [8; p.451 (1.7)],

$$\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}=\exp(2xt-yt^2), \quad ...(1.2)$$

which for y = 1, gives the generating function of Hermite polynomials, $H_n(x)$, [4] and see also [9]

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}=\exp(2xt-t^2) \quad ...(1.3)$$

By Pathan, Khan and Yasmin [6] are derived generating function of two variables (G.H.P.), $H_n(x,y)$ by relating them to the representation theory of a five dimensional Lie algebra K_5 [See [5;p.298]].

A more interesting aspect of the problem is associated with the multivariable extension of the previous families of Polynomials.

Dattoli et. al [3] introduce the p-variable Bell type polynomials [1] defined by

$$H_n^{(2,\dots,p)}(x_1,\dots,x_p)=n!\sum_{r=0}^{[n/p]}\frac{x_p^r\, H_{n-pr}^{(2,\dots,p-1)}(x_1,\dots,x_{p-1})}{r!(n-pr)!} \quad ...(1.4)$$

With generating function

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n^{(2,\dots,p)}(x_1,\dots,x_p)=\exp\left(\sum_{s=0}^{p} x_s t^s\right). \quad ...(1.5)$$

For p = 3, equation (1.5) reduces to

$$\sum_{n=0}^{\infty} H_n^{(3)}(x_1,x_2,x_3)\frac{t^n}{n!}=e^{x_1t+x_2t^2+x_3t^3}, \quad ...(1.6)$$

for $x_3 = 0$, equation (1.6) reduces to

$$\sum_{n=0}^{\infty} H_n^{(3)}(x_1,x_2)\frac{t^n}{n!}=e^{x_1t+x_2t^2}, \quad ...(1.7)$$

because

$$H_n^{(3)}(x_1,x_2,0)=H_n^{(2)}(x_1,x_2) \quad ...(1.8)$$

For $x_2 = 0$, (1.6) reduces to

$$\sum_{n=0}^{\infty} H_n^{(3)}(x_1,x_3)\frac{t^n}{n!}=e^{x_1t+x_3t^3} \quad ...(1.9)$$

and

$$(n-k)!=\frac{(-1)^k n!}{(-n)_k};0\le k\le n, \quad ...(1.10)$$

Linear Generating Functions

Theorem: Any value of parameters and variables leading to result which do not make sense are tacitly excluded then

$$\sum_{n=0}^{\infty} H_{n+k}(x,y,z)\frac{t^n}{n!}=\exp(2xt-yt^2+zt^3)H_k(x-yt+\frac{3}{2}zt^2,y-3zt,z) \quad ...(2.1)$$

Proof: Consider the following double series

$$S=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} H_{n+k}(x,y,z)\frac{t^n\nu^k}{n!k!}, \quad ...(2.2)$$

$$=\sum_{n=0}^{\infty}\frac{H_n(x,y,z)}{n!}(t+\nu)^n$$

Now using equation (1.1), we get

$$S=\exp(2xt-yt^2+zt^3)\sum_{k=0}^{\infty}\left(x-yt+\frac{3z}{2}t^2,y-3zt,z\right)\frac{\nu^k}{k!}. \quad ...(2.3)$$

Comparing the coefficient of $\frac{\nu^k}{k!}$ in (2.2) and (2.3) on both side, we get required result (2.1).

Theorem: To prove

$$\sum_{n=0}^{\infty} H_{n+k}^{(3)}(x,y,z)\frac{t^n}{n!}$$
$$=\exp(xt+yt^2+zt^3)H_k^{(3)}(x+2yt+3zt^2,\, y-3zt,\, z) \quad ...(2.4)$$

Proof: Consider the following double series

$$S=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} H_{n+k}^{(3)}(x,y,z)\frac{t^n \nu^k}{n!k!} \quad ...(2.5)$$

$$=\sum_{n=0}^{\infty}\frac{H_n^{(3)}(x,y,z)(t+\nu)^n}{n!},$$

Again using (1.6), we get

$$S=\exp(xt+yt^2+zt^3)\sum_{k=0}^{\infty} H_k^{(3)}(x+2yt+3zt^2, y+3zt, z)\frac{\nu^k}{k!} \quad ...(2.6)$$

Equating the coefficient of $\frac{\nu^k}{k!}$ in (2.5) and (2.6) on the both side, we get the required result (2.4).

Theorem: To prove

$$\sum_{n=0}^{\infty}(c)_n H_n(x,y,z)\frac{t^n}{n!}$$
$$=(1-2xt)^{-c}\sum_{r=0}^{\infty}\frac{\left(\frac{c}{3}\right)_r\left(\frac{c+1}{3}\right)_r\left(\frac{c+2}{3}\right)_r}{r!}\left[\frac{27zt^3}{(1-2xt)^3}\right]^r$$
$$\times {}_2F_0\left[\frac{c+3r}{2},\frac{c+3r+1}{2};-;\frac{-4yt^2}{(1-2xt)^2}\right] \quad ...(2.7)$$

Proof: Consider the following series

$$S=\sum_{n=0}^{\infty}(c)_n H_n(x,y,z)\frac{t^n}{n!} \quad ...(2.8)$$

Now using (1.4) for p = 3,

$$S=\sum_{n=0}^{\infty}\sum_{r=0}^{[n/3]}(c)_n\frac{z^r H_{n-3r}(x,y)\,t^n}{r!(n-3r)!}.$$

Now using series rearrangement technique and Legendre's duplication formula, we get

$$S=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{3^{3r}\left(\frac{c}{3}\right)_r\left(\frac{c+1}{3}\right)_r\left(\frac{c+2}{3}\right)_r\left(\frac{c+3r}{2}\right)_k\left(\frac{c+3r+1}{2}\right)_k}{k!r!n!}$$
$$\times 2^{2k}(c+3r+2k)_n(2x)^n(-y)^k z^r t^{n+2k+3r}$$

$$S=(1-2xt)^{-c}\sum_{r=0}^{\infty}\left(\frac{c}{3}\right)_r\left(\frac{c+1}{3}\right)_r\left(\frac{c+2}{3}\right)_r\left[\frac{27zt^3}{(1-2xt)^3}\right]^r$$

$$\times {}_2F_0\left[\frac{c+3r}{2},\frac{c+3r+1}{2};-;\frac{-4yt^2}{(1-2xt)^2}\right] \quad ...(2.9)$$

Now equating the equations (2.8) and (2.9), we get required result (2.7).

SPECIAL CASES

(i) For z = 0 in (2.1) and using (1.2), we get

$$\sum_{n=0}^{\infty} H_{n+k}(x,y)\frac{t^n}{n!}=\exp(2xt-yt^2)H_k(x-yt,y) \quad ...(3.1)$$

This is a known generating relation given by Pathan, Yasmeen and Qureshi [8, p.452 (2.4)].

Again putting y =1 in (3.1), we get

$$\sum_{n=0}^{\infty} H_{n+k}(x)\frac{t^n}{n!}=\exp(2xt-t^2)H_k(x-t) \quad ...(3.2)$$

which is a known result [9; p.197(1)].

(ii) For z = 0 in (2.4), then we get a generating relation

$$\sum_{n=0}^{\infty} H_{n+k}^{(2)}(x,y)\frac{t^n}{n!}=\exp(xt+yt^2)H_k^{(2)}(x+2yt,y) \quad ...(3.3)$$

which is a known result [8; p.452 (2.1)].

(iii) For z = 0 when r = 0, y = 1, then equation (2.7) reduces to [9, p.190(1)].

(iv) For y = 0, then (2.7) reduces to

$$\sum_{n=0}^{\infty}(c)_n H_n^{(3)}(x,z)\frac{t^n}{n!}=(1-2xt)^{-2}\,{}_3F_0\left[\begin{matrix}\frac{c}{3},\frac{c+1}{3},\frac{c+2}{3};\\ ---;\end{matrix}\frac{27zt^3}{(1-2xt)^3}\right]$$

This is new and un known result.

(v) For x = 0, then (2.7) reduces to

$$\sum_{n=0}^{\infty}(c)_n H_n^3(y,z)\frac{t^n}{n!}=\sum_{r=0}^{\infty}\left(\frac{c}{3}\right)_r\left(\frac{c+1}{3}\right)_r(27zt^3)^r$$
$$\times {}_2F_0\left[\frac{c+3r}{2},\frac{c+3r+1}{2};-;-4yt^2\right]$$

This is a new and unknown result.

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