OPTIMAL FOURTH ORDER VARIANTS OF ELLIPSE METHODS

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Abstract : This paper presents quartically convergent variants of ellipse method for the solution of scalar non-linear equation, permitting $f'(x)=0$ near the root. The methods are obtained by combining the quadratically convergent ellipse method with the false position method. Per iteration the new method requires two evaluations of the function and one evaluation of its first order derivative and therefore, has the efficiency index equal to 1.587. Several examples are given to demonstrate the efficiency and performance of the modified ellipse method. Further, numerical experiments demonstrate that the quartically convergent variants of ellipse method outperform the classical Newton's and other variants of Newton's method.

Keywords: Non-linear equations; Iterative methods; False position method; Newton's method; Ellipse method; Halley's method; Chebyshev's method; Traub-Ostrowski's method; Order of convergence

1. Introduction : Finding a solution for a non-linear scalar equation is an age-old problem. Newton's method [1-3] which converges quadratically to a simple root of $f(x)=0$, is probably the best known and most widely used algorithm. It is given by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \quad (1)$$

However, the problem with the Newton's method is that it may fail to converge in some cases if the derivative of the function is very small or zero in the vicinity of the required root. There is a class of third-order methods requiring the evaluations of second order derivative such as Halley's irrational method [1-3], Halley's rational method [1-3], Chebyshev's method [1-3] etc and are close relatives of Newton's method. The main practical difficulty associated with these methods is the evaluations of second order derivative. Many researchers developed modifications of Newton's method or Newton-like methods in a number of ways to improve the local order of convergence of Newton's method at the expense of additional evaluations of functions and/ or derivatives mostly at the point iterated by the method. All these modifications are targeted at increasing the local order of convergence with a view of increasing their efficiency index. But all these methods are variants of Newton's method and will fail miserably if $f'(x)$ is very small or zero in the vicinity of the root. Therefore, the researches of developing higher-order methods which are suitable in the problems where $f'(x)=0$ is permitted and free from second order derivative are important for practical applications.

More recently, Gupta et al. [4] have developed a family of ellipse methods given by

$$x_{n+1}=x_n\pm\frac{f(x_n)}{\sqrt{f'^2(x_n)+p^2f^2(x_n)}}, \quad (2)$$

where $p\neq 0\in\Re$ and in which $f'(x)=0$ is permitted near the root. The beauty of this method is that it converges quadratically and moreover, has the same error equation as Newton's method. Therefore, this method is an efficient alternative to Newton's method.

In this article, we further present a quartically convergent family of ellipse methods based on ellipse method (2) and linearly convergent false position method. Per iteration the method requires two evaluations of the function and one evaluation of its first order derivative. Therefore, this family is very efficient because of the improvement of its local order of convergence.

2. Development of the method

Assume the equation

$$f(x)=0, \quad (3)$$

has a simple root r which is to be found and let $x_0=r+e_0$ be our initial approximation to this root. Let $y=f(x)$ represents the graph of the function $f(x)$. Further, let

$$y_0=f(x_0), \quad (4)$$

and $x_1 = x_0 \pm \dfrac{f(x_0)}{\sqrt{f'^2(x_0)+p^2 f^2(x_0)}}$ (5)

be the quadratically convergent ellipse method [4].

Let $y_1 = f(x_1)$. If, on the same graph of $y = f(x)$, (6)

we draw a straight line through the points (x_1, y_1) and $\left\{\left(\dfrac{x_0+x_1}{2}\right), \dfrac{f(x_0)}{2}\right\}$, then equation of this straight is given by

$$y - y_1 = \frac{y_0 - 2y_1}{x_0 - x_1}(x - x_1). \quad (7)$$

If initial guess x_0 is sufficiently close to the required root r, then the intersection of a line (7) and the $x-axis$ gives a good approximation to the root. Therefore, the first approximation to the root is given by

$$x_2 = x_1 - \frac{(x_0 - x_1)}{y_0 - 2y_1} y_1. \quad (8)$$

This is a quartically convergent family of ellipse methods. In general we see that

$$x_{n+1} = x_n \pm \frac{f(x_n)}{\sqrt{f'^2(x_n)+p^2 f^2(x_n)}} \left\{ \frac{f(x_n) - f\left(x_n \pm \dfrac{f(x_n)}{\sqrt{f'^2(x_n)+p^2 f^2(x_n)}}\right)}{f(x_n) - 2f\left(x_n \pm \dfrac{f(x_n)}{\sqrt{f'^2(x_n)+p^2 f^2(x_n)}}\right)} \right\},$$

(9)

where the positive sign is taken if $x_0 < r$ and negative sign is taken if $x_0 > r$.

It is interesting to note that by ignoring the term in p, equation (9) gives the famous quartically convergent variant of Newton's method namely Traub-Ostrowski's formula [1-3]. For the formula (9), we prove the following convergence theorem:

3. Convergence analysis

Theorem 3.1 Suppose $f(x)$ is sufficiently differentiable function in a neighborhood of a simple root r and that x_0 is close to r, then the iteration scheme (9) has fourth-order convergence and satisfies the following error equation:

$$e_{n+1} = \left\{A_2^{\,3} - A_2 A_3 - \frac{1}{2} p^2 A_2\right\} e_n^{\,4} + O\left(e_n^{\,5}\right).$$

(10)

Proof: Since $f(x)$ is sufficiently differentiable, expanding $f(x_n)$ and $f'(x_n)$ about $x = r$ by Taylor's expansion, we have

$$f(x_n) = f'(r)\left[e_n + A_2 e_n^{\,2} + A_3 e_n^{\,3} + A_4 e_n^{\,4} + A_5 e_n^{\,5} + O\left(e_n^{\,6}\right)\right],$$

(11)

and $f'(x_n) = f'(r)\left[1 + 2A_2 e_n + 3A_3 e_n^{\,2} + 4A_4 e_n^{\,3} + 5A_5 e_n^{\,4} + O\left(e_n^{\,5}\right)\right]$

, (12)

where $e_n = |\,x_n - r\,|$ and

$$A_k = \frac{1}{k!}\frac{f^k(r)}{f'(r)}, k = 2,3,...$$

Using (11) and (12), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - A_2 e_n^{\,2} - 2\left(A_3 - A_2^2\right)e_n^{\,3} - \left(3A_4 - 7A_2A_3 + 4A_2^3\right)e_n^{\,4} + O\left(e_n^{\,5}\right)$$

(13)

Therefore,

$$\frac{f(x_n)}{\sqrt{f'^2(x_n)+p^2 f^2(x_n)}} = \frac{f(x_n)}{f'(x_n)\sqrt{1+p^2\left\{\dfrac{f(x_n)}{f'(x_n)}\right\}^2}},$$

$$= e_n - A_2 e_n^{\,2} - \left\{\frac{1}{2}p^2 - 2\left(A_2^{\,2} - A_3\right)\right\}e_n^{\,3} + \left\{\frac{3}{2}p^2 A_2 + 7A_2A_3 - 4A_2^{\,3} - 3A_4\right\}e_n^{\,4} + O\left(e_n^{\,5}\right)$$

. (14)

Furthermore

$$f\left(x_n \pm \frac{f(x_n)}{\sqrt{f'^2(x_n)+p^2 f^2(x_n)}}\right) = f'(r)\left[\begin{array}{l} A_2 e_n^{\,2} + \left\{\dfrac{1}{2}p^2 - 2\left(A_2^{\,2} - A_3\right)\right\}e_n^{\,3} \\ -\left\{\dfrac{3}{2}p^2 A_2 + 7A_2A_3 - 5A_2^{\,3} - 3A_4\right\}e_n^{\,4} + O\left(e_n^{\,5}\right) \end{array}\right]$$

. (15)

and

$$f(x_n) - 2f\left(x_n \pm \frac{f(x_n)}{\sqrt{f'^2(x_n)+p^2 f^2(x_n)}}\right) = f'(r)\left[\begin{array}{l} e_n - A_2 e_n^{\,2} + \left\{4A_2^{\,2} - 3A_3 - p^2\right\}e_n^{\,3} \\ + \left\{3p^2 A_2 - 10A_2^{\,3} - 5A_4 + 14A_2A_3\right\} + O\left(e_n^{\,5}\right) \end{array}\right]$$

. (16)

Using (15) and (16), we have

$$\frac{f(x_n)-f\left(x_n \pm \dfrac{f(x_n)}{\sqrt{f'^2(x_n)+p^2 f^2(x_n)}}\right)}{f(x_n)-2f\left(x_n \pm \dfrac{f(x_n)}{\sqrt{f'^2(x_n)+p^2 f^2(x_n)}}\right)} = 1 + A_2 e_n + \left\{\frac{1}{2}p^2 - A_2^2 + 2A_3\right\}e_n^2 + \{3A_4 - 2A_2 A_3\}e_n^3 + O(e_n^4)$$

(17)

Using (14) and (17) in (9), we get the error equation as

$$e_{n+1} = \left\{A_2^3 - A_2 A_3 - \frac{1}{2}p^2 A_2\right\}e_n^4 + O(e_n^5). \quad (18)$$

This completes the proof of the theorem.

4. Numerical examples

We employ the quartically convergent variant of ellipse method (QVEM) to solve some non-linear equations and compare this with Newton's method (NM), Halley's method (HM), Chebyshev's method (CM) and Traub-Ostrowski's method (TOM). The formula (9) is tested for $p = \frac{1}{2}$ and the results are summarized in Table 2. We use $\in = 10^{-15}$ as tolerance. Computations have been performed using C^{++} in double precision arithmetic. The following stopping criteria are used for computer programs:

(i) $|x_{n+1} - x_n| < \in$, (ii) $|f(x_{n+1})| < \in$.

Table 1

Test functions

$f(x)$	Root (r)
$f_1 = \arctan(x)$	0.00000000000
$f_2 = e^{x^2+7x-30} - 1$	3.00000000000
$f_3 = (x-1)^6 - 1$	2.00000000000
$f_4 = x^3 + 4x^2 - 10$	1.36522996425
$f_5 = \cos(x) - x$	0.73908513784
$f_6 = \ln(x)$	1.000000000000

Table 2

Performance of the methods (Number of iterations required)

D:-Divergent, F:-Fails

$f(x)$	x_0	NM	HM	CM	TO-M	QVE-M
f_1	-2.0	D	4	D	5	3
	2.0	D	4	D	5	3
f_2	2.0	D	8	D	D	2
	2.5	D	5	D	D	6
	2.8	15	4	D	5	4
	3.5	11	6	7	5	5
f_3	1.1	58	9	89*	25	3
	3.0	8	4	5	4	4
f_4	0.0	F	F	F	F	3
	0.1	9	5	74	4	2
	2.0	4	3	3	2	2
f_5	-1.0	7	5	D	9	3
	2.0	5	3	3	2	3
f_6	3.0	D	3	4	D	3

* (Chebyshev's method converges to the root zero)

5. Conclusions

The family of fourth-order variants of ellipse method is the main findings of the present work. The presented results in Table 2 indicate that the presented quartically convergent variants of ellipse method improve the computational efficiency of the ellipse method. Moreover, these methods do not require the evaluation of second order derivative and do not fail even if the derivative of the function is either zero or very small in the vicinity of the root. Finally, we conclude that the methods presented in this article are competitive with other well-known methods, namely, Newton's, Halley's, Chebyshev's, Traub-Ostrowski's method etc. and have efficiency index equal to $\sqrt[3]{4} \cong 1.587$ which is better than the one of Newton's method $\sqrt[2]{2} \cong 1414$ and Halley's method $\sqrt[3]{3} \cong 1.442$.

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