COEFFICIENT INEQUALITY FOR A COMBINED SUBCLASS OF STARLIKE AND INVERSE STARLIKE CLASSES OF ANALYTIC FUNCTIONS

BHARAT GOYAL, DIMPLE RANI

Abstract: Here we will discuss a newly constructed class of analytic functions and its subclasses by which we will be obtaining sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes and subclasses.

Keywords: Univalent functions, Starlike functions, Inverse Starlike functions and bounded functions.

Introduction : Let $\mathcal{A}$ denote the class of functions of the form

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ (1.1)

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([7], [8]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$, Fekete and Szegö[9] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if\ \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \leq \mu \leq 1; \\ 4\mu - 3, if \mu \geq 1. \end{cases}$$

(1.2)

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (See Chhichra[1], Babalola[6]).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

$\in \mathcal{A}$ and satisfying the condition

$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}.$ (1.3)

We denote by $\mathcal{K}$, the class of univalent convex functions $h(z) = z + \sum_{n=2}^{\infty} c_n z^n$, $z \in \mathcal{A}$ and satisfying the condition

$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}.$ (1.4)

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in \mathbb{E}.$ (1.5)

The class of close to convex functions is denoted by C and was introduced by Kaplan [3] and it was shown by him that all close to convex functions are univalent.

$S^*(A,B) = \left\{f(z) \in \mathcal{A}; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\}$ (1.6)

$\mathcal{K}(A,B) = \left\{f(z) \in \mathcal{A}; \frac{(zf'(z))'}{f'(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\}$ (1.7)

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}(A,B)$ is a subclass of $\mathcal{K}$.

We introduce a new subclass as

$\left\{\mathbf{f(z)} \in \mathcal{A}; (1-\alpha)\frac{zf(z)}{2\int_0^z f(z)\,dz} + \alpha\frac{zf'(z)}{f(z)} \prec \left(\frac{\mathbf{1+z}}{\mathbf{1-z}}\right)^{\boldsymbol{\delta}}; \mathbf{z} \in \mathbb{E}\right\}$ and we will denote this class as $KS^{*-1}(\alpha,\delta)$.

It is to be noted that

- $KS^{*-1}(\alpha, 1) = KS^{*-1}$
- $KS^{*-1}(0,\delta) = S^{*-1}(\delta)$
- $KS^{*-1}(1,\delta) = S^*(\delta)$
- $KS^{*-1}(0,1) = S^{*-1}$
- $KS^{*-1}(1,1) = S^*$

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z)); z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form

$w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1.$ (1.8)

It is known that $|d_1| \leq 1, |d_2| \leq 1 - |d_1|^2$ (1.9)

Preliminary Lemmas: For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz}\right)$ so that

$$\frac{1+Aw(z)}{1+Bw(z)} = 1 + (A-B)c_1 z + (A-B)(c_2 - Bc_1^2)z^2 + --- \quad (2.1)$$

3.Main Results

THEOREM 3.1: Let $f(z) \in KS^{*-1}(\alpha, \delta)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{4\delta^2(4\alpha^2+18\alpha+5)}{(3\alpha+1)(2\alpha+1)^2} - \frac{36\mu\delta^2}{(2\alpha+1)^2} \ \ if \mu \leq \frac{\alpha^2(4\delta-4)+\alpha(18\delta-4)+(5\delta-1)}{9\delta(3\alpha+1)} \\ \frac{4\delta}{3\alpha+1} if \frac{\alpha^2(4\delta-4)+\alpha(18\delta-4)+(5\delta-1)}{9\delta(3\alpha+1)} < \mu < \frac{\alpha^2(4\delta+4)+\alpha(18\delta+4)+(5\delta+1)}{9\delta(3\alpha+1)} \\ \frac{36\mu\delta^2}{(2\alpha+1)^2} - \frac{4\delta^2(4\alpha^2+18\alpha+5)}{(3\alpha+1)(2\alpha+1)^2} \ \ if \ \mu \geq \frac{\alpha^2(4\delta+4)+\alpha(18\delta+4)+(5\delta+1)}{9\delta(3\alpha+1)} \end{cases}$$

The results are sharp.

Proof: By definition of $f(z) \in KS^{*-1}(\alpha, \delta)$, we have

$$(1-\alpha)\frac{zf(z)}{2\int_0^z f(z)\,dz} + \alpha\frac{zf'(z)}{f(z)} = \left(\frac{1+w(z)}{1-w(z)}\right)^{\delta}; w(z) \in \mathcal{U}. \quad (3.4)$$

Expanding the series (3.4), we get

$$\alpha\left(z^3 + z^4\left(\frac{2}{3}a_2 + 2a_2\right) + z^5\left(\frac{1}{2}a_3 + \frac{4}{3}{a_2}^2 + 3a_3\right) + --------\right) + (1-\alpha)z(z^2 + 2a_2 z^3 + {a_2}^2 z^4 + 2a_3 z^4 + ------) = \left(z^3 + z^4\left(\frac{2}{3}a_2 + a_2\right) + z^5\left(\frac{1}{2}a_3 + \frac{2}{3}{a_2}^2 + a_3\right) + -----\right)\left(1 + 2\delta c_1 z + 2\delta(c_2 + {c_1}^2)z^2 + \frac{\delta(\delta-1)}{2}4({c_1}^2)z^2 --------\right) \quad (3.5)$$

Identifying terms in (3.5), we get

$$a_2 = \frac{6\delta c_1}{2\alpha+1}, \quad \alpha \neq \frac{-1}{2} \quad (3.6)$$

$$a_3 = \frac{4}{3\alpha+1}\left(\frac{\delta^2{c_1}^2(4\alpha^2+18\alpha+5)}{(2\alpha+1)^2} + \delta c_2\right) \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$a_3 - \mu a_2^2 = {c_1}^2\left(\frac{\delta^2(16\alpha^2+72\alpha+20) - 36\mu\delta^2(3\alpha+1)}{(3\alpha+1)(2\alpha+1)^2}\right) + \frac{4\delta c_2}{(3\alpha+1)} \quad (3.8)$$

Taking absolute value, (3.8) can be rewritten as

$$|a_3 - \mu a_2^2| \leq |c_1|^2\left(\left|\frac{\delta^2(16\alpha^2+72\alpha+20) - 36\mu\delta^2(3\alpha+1)}{(3\alpha+1)(2\alpha+1)^2}\right| - \frac{4\delta}{(3\alpha+1)}\right) + \frac{4\delta}{(3\alpha+1)} \quad (3.9)$$

Case I: $\mu \geq \frac{(4\alpha^2+18\alpha+5)}{9(3\alpha+1)}$. (3.9) can be rewritten as

$$|a_3 - \mu a_2^2| \leq |c_1|^2\left(\frac{-4\delta^2(4\alpha^2+18\alpha+5)}{(3\alpha+1)(2\alpha+1)^2} - \frac{4\delta}{3\alpha+1} + \frac{36\mu\delta^2}{(2\alpha+1)^2}\right) + \frac{4\delta}{(3\alpha+1)} \quad (3.10)$$

Subcase I (a): $\mu \leq \frac{\alpha^2(4\alpha+4)+\alpha(18\delta+4)+5\delta+1}{9\delta(3\alpha+1)}$ Using (1.8), (3.10) becomes

$$|a_3 - \mu a_2^2| \leq \frac{36\mu\delta^2}{(2\alpha+1)^2} - \frac{4\delta^2(4\alpha^2+18\alpha+5)}{(3\alpha+1)(2\alpha+1)^2} \quad (3.11)$$

Subcase I (b): $\mu \leq \frac{\alpha^2(4\alpha+4)+\alpha(18\delta+4)+5\delta+1}{9\delta(3\alpha+1)}$ We obtain from (3.10)

$$|a_3 - \mu a_2^2| \leq \frac{4\delta}{(3\alpha+1)} \quad (3.12)$$

Case II: $\mu < \frac{(4\alpha^2+18\alpha+5)}{9(3\alpha+1)}$

Preceding as in case I, we get

$$|a_3 - \mu a_2^2| \leq |c_1|^2\left(\frac{4\delta^2(4\alpha^2+18\alpha+5)}{(3\alpha+1)(2\alpha+1)^2} - \frac{4\delta}{3\alpha+1} - \frac{36\mu\delta^2}{(2\alpha+1)^2}\right) + \frac{4\delta}{(3\alpha+1)} \quad (3.13)$$

Subcase II (a): $\mu \leq \frac{\alpha^2(4\alpha-4)+\alpha(18\delta-4)+5\delta+1}{9\delta(3\alpha+1)}$

(3.13) takes the form $|a_3 - \mu a_2^2| \leq \frac{4\delta^2(4\alpha^2+18\alpha+5)}{(3\alpha+1)(2\alpha+1)^2} - \frac{36\mu\delta^2}{(2\alpha+1)^2}$

Subcase II (b): $\frac{\alpha^2(4\alpha-4)+\alpha(18\delta-4)+5\delta+1}{9\delta(3\alpha+1)} < \mu < \frac{(4\alpha^2+18\alpha+5)}{9(3\alpha+1)}$

Preceding as in subcase I (a), we get

$|a_3 - \mu a_2^2| \leq \frac{4\delta}{3\alpha+1}$ (3.14)

Thus the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by

$$f_1(z) = z\left[1 + \frac{4\delta^3(4\alpha^2+18\alpha+5)^2 - 3(3\alpha+1)^2(2\alpha+1)^3}{\delta(3\alpha+1)(2\alpha+1)^2(4\alpha^2+18\alpha+5)} z\right]^{\frac{4\delta^3(4\alpha^2+18\alpha+5)^2}{4\delta^3(4\alpha^2+18\alpha+5)^2 - 3(3\alpha+1)^2(2\alpha+1)^3}}$$

Extremal function for (3.2) is defined by $f_2(z) = z(1+\delta z)^{\frac{4}{3\alpha+1}}$

Corollary 3.2: Putting $\delta = 1$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{4(4\alpha^2+18\alpha+5)}{(3\alpha+1)(2\alpha+1)^2} - \frac{36\mu}{(2\alpha+1)^2} & if \mu \leq \frac{14\alpha+4}{9(3\alpha+1)} \\ \frac{4}{3\alpha+1} \; if \; \frac{14\alpha+4}{9(3\alpha+1)} < \mu < \frac{8\alpha^2+22\alpha+6}{9(3\alpha+1)} & \\ \frac{36\mu}{(2\alpha+1)^2} - \frac{4(4\alpha^2+18\alpha+5)}{(3\alpha+1)(2\alpha+1)^2} & if \; \mu \geq \frac{8\alpha^2+22\alpha+6}{9(3\alpha+1)} \end{cases}$$

These are the results of the class KS^{*-1}.

Corollary 3.3: Putting $\alpha = 0$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} (20-36\mu)\delta^2 & if \mu \leq \frac{5\delta-1}{9\delta} \\ 4\delta & \frac{5\delta-1}{9\delta} < \mu < \frac{5\delta+1}{9\delta} \\ (36\mu-20)\delta^2 & if \; \mu \geq \frac{5\delta+1}{9\delta} \end{cases}$$

These are the results of the class $S^{*-1}(\delta)$.

Corollary 3.4: Putting $\alpha = 1$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} (3-4\mu)\delta^2 & if \mu \leq \frac{3\delta-1}{4\delta} \\ \delta & \frac{3\delta-1}{4\delta} < \mu < \frac{3\delta+1}{4\delta} \\ (4\mu-3)\delta^2 & if \; \mu \geq \frac{3\delta+1}{4\delta} \end{cases}$$

These are the results of the class $S^{*}(\delta)$.

Corollary 3.5: Putting $\alpha = 0, \delta = 1$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 20-36\mu & if \mu \leq \frac{4}{9} \\ 4 & \frac{4}{9} < \mu < \frac{2}{3} \\ 36\mu-20 & if \; \mu \geq \frac{2}{3} \end{cases}$$

These are the results of the class S^{*-1}.

Corollary 3.6: Putting $\alpha = 1, \delta = 1$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3-4\mu & if \mu \leq \frac{1}{2} \\ 1 & \frac{1}{2} < \mu < 1 \\ 4\mu-3 & if \; \mu \geq 1 \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent star like functions.

References:

1. Chichra, P. N., "New subclasses of the class of close-to-convex functions", Procedure of American Mathematical Society, 62, (1977), 37-43.
2. *S. Sudha, M. Satheeshkumar, V. Kanniga,* on Complete, Strong, Regular Fuzzy Graphs; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 2 Issue 2 (2013), Pg 650-654
3. Goel, R. M. and Mehrok, B. S., "A subclass of univalent functions", Houston Journal of Mathematics, 8, (1982), 343-357.
4. Kaplan, W., "Close-to-convex schlicht functions", Michigan Mathematical Journal, 1, (1952), 169-185.
5. Keogh, S. R. and Merkes, E. R., "A Coefficient inequality for certain classes of analytic functions", Procedure of American Mathematical Society, 20, (1989), 8-12.
6. K. Löwner, "Uber monotone Matrixfunktionen", Math. Z., 38, (1934), 177-216.
7. V.Chandrasekar ,M.Sathiyamoorthy, Oscillatory Behaviour of Solutions of Certain Type; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 3 Issue 1 (2014), Pg 201-207
8. Kunle Oladeji Babalola, "The fifth and sixth coefficients of α-close-to-convex functions", Kragujevac J. Math., 32, (2009), 5-12.
9. L. Bieberbach, "Uber einige Extremal probleme im Gebiete der konformen Abbildung", Math. Ann., 77(1916), 153-172.
10. M. Muthukumari, A. Nagarajan, M. Murugalingam, Fuzzification of Filters; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 3 Issue 2 (2014), Pg 669-671
11. L. Bieberbach, "Uber die Koeffizientem derjenigen Potenzreihen, welche eine schlithe Abbildung des Einheitskreises vermitteln", Preuss. Akad. Wiss. Sitzungsb., (1916), 940-955.
12. M. Fekete and G. Szegö, "Eine Bemerkung über ungerade schlichte Funktionen", J. London Math. Soc. 8 (1933), 85-89.
13. R. M. Goel and B. S.Mehrok, "A coefficient inequality for a subclass of closeto-convex functions", Serdica Bul. Math. Pubs., 15, (1989), 327-335.
14. *Rekha V. Joshi,* Multiplication Semigroups With Identity; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 2 Issue 2 (2013), Pg 302-307
15. Z. Nehari, "Conformal Mapping", McGraw-Hill, Comp., Inc., New York, (1952).

Bharat Goyal
Khalsa College, Patiala
(E-Mail: Bhartu89@Gmail.Com)
Dimple Rani (Corresponding Author)
Khalsa College, Patiala
(Email: Dimplegoyal938@Gmail.Com)