ADDTION OF TWO DIMENSIONAL RANDOM VARIABLES USING CONTINUOUS DISTRIBUTIONS

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Abstract : In this paper sum properties of two dimensional random variables of continuous uniform- negative exponential distribution has been presented. Using the Jacobean transformation technique of these random variables the probability density function of distribution was obtained. The various distributional properties like as expected value, standard deviation, Moment generating function, etc., were discussed.

Keywords: Two dimensional random variables, Continuous Uniform distribution, Moment generating function.

Introduction : robability distributions gained lot of importance due to their ready applicability for analyzing several data sets arising at space and biological experiments, Agricultural experiments, Business analytics, Data mining, Geo-informatics, Stochastic modeling etc., The distributions are broadly classified into discrete and continuous distributions. Continuous distributions are generally amenable to more aligent mathematical treatment than are discrete distributions. This makes them especially useful as approximations to discrete distributions. The continuous distributions are used in the construction of the models and in applying statistical techniques. Continuous distributions have been used as approximations to discrete distributions (N.L. Johnson, Samuel Kotz, N.Balakrishnan (2004)).

In the last five decades generalizations of continuous distributions are developed with various considerations. The common consideration is generalization of a distribution is introducing other parameters or ascribing a probability distribution to one or more parameters. In this fashion compound distributions were developed. Away from this another direction of generalization is considering mixture of distributions with weights to component densities. These mixtures of distributions are initially considered with the same population densities. Later they extended to the mixture of heterogeneous densities. Another generalization considered in these continuous distributions are Gram-charier series expansions of distributions using normal as parent distribution. To accommodate a wider class of distributions one area of research is considering personae an system of distributions.

However in some practical situations, the variety under study may be a sum of two or more random variables. Much work has been reported in literature regarding the distribution of the sum of random variables by considering that the varieties under study are from the same homogenous population or the variation may be with reference to the parameters (but the functional form remains the same). In some other practical situations the random variable under study may be a sum of two different types of populations. For example in Manpower Modeling the complete length of service of an employee in the organization is the sum of two random time periods one for probation (temporary) period another for committed (permanent) period. Less work has been reported regarding the distribution of the sum of different random variables with different populations especially with uniform and exponential populations. This paper is an attempt to fill the gap in this area of research by developing and analyzing the Additive Uniform Exponential Distribution. This distribution is much useful in analyzing many data sets arising at Communication Systems, Biological Systems, Signal Processing, Inventory Control, etc.,

Using the Jacobean transformation of random variables the probability density function of the distribution is obtained. The various distributional properties like, mean, variance, moment generating function, recurrence relation of moments, Skewness, kurtosis, etc., are discussed. The recurrence relations of the distribution are also studied. Some inferential aspects of this distribution are presented. This distribution includes uniform and exponential distributions as particular cases for limiting values of the parameters.

2. PROBABILITY DENSITY FUNCTION:

Let X be a continuous random variable follows uniform distribution with parameter 'β 'in the domain $(0, \beta)$ and Y be another continuous random variable follows exponential distribution with parameter 'θ' in the domain (0, ∞) such that X and Y are independent.

The probability density function of X is

$f(x) = \frac{1}{\beta}; 0 \leq X \leq \beta; \quad \beta > 0$ (1)

The probability density function of Y is

$f(y) = \theta e^{-\theta y} ; 0 \leq Y < \infty$ (2)

Using the transformation of random variables $V = Y$ and $U = X + Y$, the probability density function of U is obtained.

For $0 \leq U \leq \beta$; the probability density of U is

$f(u) = \int_0^u g(u\, v)\, dv = \int_0^u \frac{1}{\beta} \theta e^{-\theta v} dv$

$f(u) = \frac{1}{\beta}\left[1 - e^{-\theta u}\right]; 0 \leq U \leq \beta$ (3)

For $\beta \leq U < \infty$; the probability density of U is

$f(u) = \int_{u-\beta}^{u} g(u\, v)\, dv = \int_{u-\beta}^{u} \frac{1}{\beta} \theta e^{-\theta v} dv$

Then,

$f(u) = \frac{e^{-\theta u}}{\beta}\left[e^{\beta\theta} - 1\right] ; \beta \leq U < \infty$(4)

A continuous random variable U is said to follow Additive Uniform Exponential Distribution if its probability density function is of the form

$f(u) = \frac{1}{\beta}\left[1 - e^{-\theta u}\right] ; 0 \leq U \leq \beta$

$= \frac{e^{-\theta u}}{\beta}\left[e^{\beta\theta} - 1\right]; \beta \leq U < \infty$(5)

Where β and θ are the parameters of the distribution, $\beta > 0; \theta > 0$

Since, $\lim_{\theta \to \infty} f(u) = \frac{1}{\beta}$ this distribution includes uniform distribution as a particular case if $\theta \to \infty$ and $\beta \to 0$ this distribution includes exponential as a particular case $\lim_{\beta \to 0} f(u) = \theta e^{-\theta u}; u > 0$

Distributional Properties:

It is interesting to note that this distribution has explicit functional form for the distribution function

For $0 \leq U \leq \beta$;

$F_U(\text{u}) = \int_0^u \frac{1}{\beta}\left(1 - e^{-\theta t}\right) dt$

$= \frac{u}{\beta} + \frac{e^{-\theta u}}{\beta\theta} - \frac{1}{\beta\theta}$ (6)

And for $\beta \leq U < \infty$;

$F_U(\text{u}) = \int_0^\beta f(t)\, dt + \int_\beta^u f(t)\, dt$

$= \left[1 + \frac{e^{-\beta\theta}}{\beta\theta} - \frac{1}{\beta\theta}\right] + \int_\beta^u \left(\frac{e^{\beta\theta}-1}{\beta}\right) dt$

$= 1 + \frac{e^{-\theta u}}{\beta\theta}\left[1 - e^{\beta\theta}\right]$ (7)

The mean of the distribution is

$E(U) = \int_0^\beta \frac{u}{a}\left[1 - e^{-\theta u}\right] du + \int_a^\infty \frac{u e^{-\theta u}}{\beta}\left[e^{\beta\theta} - 1\right] du$

$= \frac{\beta}{2} + \frac{1}{\theta}$ (8)

The moment generating function is

$M_U(t) = \int_0^\beta \frac{e^{tu}}{\beta}\left[1 - e^{-\theta u}\right] du + \int_\beta^\infty e^{tu}\left(\frac{e^{\beta\theta}-1}{\beta}\right) du$

On simplification we get

$M_U(t) = \frac{\theta(e^{\beta t}-1)}{\beta t(\theta - t)} \; ; |t| < 1$ (9)

The characteristic function of the distribution is

$\emptyset_U(t) = \int_0^\beta \frac{e^{itu}}{\beta}\left[1 - e^{-\theta u}\right] du + \int_\beta^\infty e^{itu}\left(\frac{e^{\beta\theta}-1}{\beta}\right) du$

$= \frac{\theta(e^{\beta it}-1)}{\beta it(\theta - it)}; \quad t > 0$ (10)

For obtaining the recurrence relation between the moments consider k^{th} moment about the origin

$\mu_k^1 = E\left(U^k\right) = \int_0^\beta \frac{u^k}{\beta}\left[1 - e^{-\theta u}\right] du + \int_\beta^\infty u^k e^{-\theta u}\left(\frac{e^{\beta\theta}-1}{\beta}\right) du$ (11)

Differentiating (11) with respect to u and after simplification we get

$\mu_k^1 = \left[\frac{k}{\theta}\right]\mu_{k-1}^1 + \frac{\beta^{k+1}}{\beta(k+1)}$ (12)

Substituting the values of 'k' as 1, 2, 3, 4 in equation (12) we get the first four moments about the origin of the distribution

The first raw moment is

$\mu_1^1 = \frac{1}{\theta} + \frac{\beta}{2}$ (13)

The second raw moment is

$\mu_2^1 = \frac{\beta}{\theta} + \frac{2}{\theta^2} + \frac{\beta^2}{3}$ (14)

The third raw moment is

$\mu_3^1 = \frac{\beta^3}{4} + \frac{\beta^2}{\theta} + \frac{3\beta}{\theta^2} + \frac{6}{\theta^3}$ (15)

The fourth raw moment is

$\mu_4^1 = \frac{\beta^4}{5} + \frac{\beta^3}{\theta} + \frac{4\beta^2}{\theta^2} + \frac{12\beta}{\theta^3} + \frac{24}{\theta^4}$ (16)

The recurrence relation between the central moments is obtained by the equation

$$\mu_k = E[(u - \mu)^k]$$

$= \int_0^\beta \frac{(u-\mu)^k}{\beta}\left[1 - e^{-\theta u}\right] du + \int_\beta^\infty (u - \mu)^k e^{-\theta u}\left(\frac{e^{\beta\theta}-1}{\beta}\right) du$

On simplification we get

$\mu_k = \left(\frac{k}{\theta}\right) \mu_{k-1} + \frac{1}{\beta(k+1)}\left[\left(\frac{\beta}{2} - \frac{1}{\theta}\right)^{k+1} + (-1)^k \left(\frac{\beta}{2} + \frac{1}{\theta}\right)^{k+1}\right]$ (17)

Substituting the values of 'k' as 2, 3, and 4 in equation (17) we get the central moments of the distribution.

The second central moment (variance) is

$\mu_2 = \frac{1}{\theta^2} + \frac{\beta^2}{12}$ (18)

The third central moment is

$\mu_3 = \frac{2}{\theta^2}$ (19)

The fourth central moment is

$\mu_3 = \frac{\beta}{\theta^4} + \frac{\beta^4}{80} + \frac{\beta^2}{\theta^2}$ (20)

The skewness of the distribution is

$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{6912}{[12+\beta^2\theta^2]^3}$ (21)

The kurtosis of the distribution is

$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{9[\beta^2\theta^2(40+\beta^2\theta^2)+720]}{5[\beta^2\theta^2+12]^2}$ (22)

The Hazard rate of the distribution is

$h(t) = \frac{f(t)}{1-F(t)} = \frac{\theta(1-e^{-\theta t})}{\theta(\beta-t)-e^{-\theta t}+1}$; 0 < t ≤ β

$=\theta$; $\beta \le t < \infty$ (23)

The characteristic function of the sum of two independent Additive Uniform Exponential Distribution variates is not in the form of characteristic function of the distribution, hence additive property does not holds good for this distribution.

Distribution of the square of the variates:

Let $'U'$ follows additive Uniform Exponential distribution with parameters 'β' and 'θ' whose probability density function is as given in equation (5)

Let $Y = U^2$, then

$G_Y(y) = P(U^2 \le y) = P(U \le \sqrt{y})$. Since there is no negative value for Y.

This equals to $F(\sqrt{y})$ substituting the values of the distribution from the equations (6) and (7)

Distribution of $Y = U^2$ is

$G_Y(y) = \frac{\sqrt{y}}{\beta} + \frac{e^{-\theta\sqrt{y}}}{\beta\theta} - \frac{1}{\beta\theta}$; $Y \le \beta$

$= \left[\frac{e^{\beta\theta}-1}{\beta\theta}\right] e^{-\theta\sqrt{y}} + \frac{1}{\beta\theta}\left[1 - e^{-\beta\theta}\right]$

; $\beta \le Y < \infty$ (24)

Differentiating equation (24) with respect to $'y'$ we get the probability density function of the square of Additive Uniform Exponential Distribution variate

$g_Y(y) = \frac{1}{2\beta\sqrt{y}}\left[1 - \frac{e^{-\theta\sqrt{y}}}{\theta}\right]$; $Y \le \beta^2$

$= \frac{1}{2\sqrt{y}}\left[\frac{e^{\beta\theta}-1}{\beta\theta}\right] e^{-\theta\sqrt{y}}$ $\beta^2 \le Y < \infty$ (25)

Conclusions:

In the concept of manpower modeling several authors have approximated the complete length of service distribution by different distributions. Silock (1954) and Bartholomew (1974) have considered that complete length of service of an employee in the organization follows exponential distribution. Rangarao.V (1994) modified the manpower models given by Silock (1954) and Bartholomew (1974) by assuming that complete length of service of an employee in an organization follows the right truncated exponential distribution. Prakasharao.V.V.S (1997) developed manpower models with truncated compound beta distribution. In all these papers they assumed that the employee behavior throughout the period is homogeneous. M.P Ramayya (2005) has applied the merged exponential distribution for modeling the complete length of service of an employee in the organization. However, the complete length of service of an employee varies accordingly to the behavioral factors namely; semi committed and committed states (temporary and permanent) Mc Clean(1976).Generally an employee during semi committed state will have different rate of leaving and duration of stay during this period during this period service can be characterized by uniform distribution. During the committed state, the rate of leaving is stabilized and constant, the duration of stay during this period can be characterized by exponential distribution. Therefore the total duration that an employee stays in an organization is a sum of two random variables associated with semi committed and committed states of the employee. Hence it is possible to develop a manpower model assuming that complete length of service of an employee in the organization is a random variable which follows additive uniform- negative exponential distribution. This distribution has a tremendous potential in analyzing several factors like the probability that an employee surviving in the organization, the rate of labor wastage etc.

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