CONSTRUCTION OF MULTISCALING FUNCTIONS USING MATRIX POLYNOMIALS

MUBEEN.M , NARAYANAN V

Abstract : A compactly supported orthogonal wavelet is symmetric only if it is the well known Haar Wavelet function. Theory of Multiwavelets assumes significance since it offers orthogonal, compact bases without losing symmetry. The properties of a Multiwavelet directly depends on the corresponding Multiscaling Function. Multiscaling Function is characterized by a unique symbol function, which is a matrix polynomial in complex exponential. A matrix polynomial can be constructed from its spectral data. We have obtained the necessary as well as sufficient conditions a spectral data must possess so that the corresponding matrix polynomial is the symbol function of a multiscaling function.

Introduction : Wavelet analysis is widely used in different areas of science and engineering. The task of finding a suitable basis to represent the data effectively with lower risk arises in different types of theoretical as well as practical problems. Wavelet Analysis provides better algorithms for such problems. Bases for the class of square integrable functions $L^2(R)$ can be constructed based on the concept of Multi Resolution Analysis (MRA), i.e. approximating the given data at different resolutions. An MRA can be defined as follows.

Definition 1.1 (Keinert [2] p.10). An MRA of $L^2(R)$ is a doubly infinite nested sequence of subspaces of L^2 $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots$ with properties

1. $\cup_n V_n$ is dense in L^2
2. $\cap_n V_n = 0$
3. $f(x) \in V_n \Leftrightarrow f(mx) \in V_{n+1}$ for all $n \in Z$. Here $m \in R$ is the dilation factor which is usually taken as 2.
4. $f(x) \in V_n \Leftrightarrow f(x - m^{-n}k) \in V_n$ for all $n, k \in Z$
5. There exists a function $\phi \in L^2$, $\phi : R \to C$ so that $\{\phi(x-k) : k \in Z\}$ form a basis for V_0

Then we say say that ϕ generates the MRA.

A function $\phi \in L^2$ generates an MRA if it satisfies certain properties, of them the key property is that it satisfies the scaling equation which is given as follows.

Definition 1.2 (Keinert [2] p.3). A function $\phi : R \to C$ is called a scaling function or refinable function if it satisfies the scaling equation

$$\varphi(x) = \sum_{k=0}^{l} h_k \varphi(mx - k) \qquad (1.1)$$

where $l \in N$, $h_k \in C$ and $m \in R$ is the dilation factor. Usually we take the value for the dilation factor m as 2.

A class of functions constructed using ϕ called wavelet functions acts as a basis for $L^2(R)$. The desirable properties of a good basis can be achieved by a suitable choice of ϕ. The desirable properties of ϕ include compact supportedness, symmetricity, orthogonality, higher approximation order etc. It is not possible to construct a compactly supported, orthogonal and symmetric wavelet function other than Haar wavelet [8]. But if we increase the multiplicity, i.e. generate MRA with more than one scaling function, then it is possible to construct a wavelet basis which is compactly supported, symmetric and orthogonal [10]. Even though the increase in multiplicity increases computational as well as theoretical complexity, we have several advantages in multiwavelets. The function vector $\Phi \in L^2$, $\Phi : R \to C^n$ so that $\{\Phi_i(x-k) : 1 \le i \le n, k \in Z\}$ form a stable basis for V_0 will be called a multiscaling function and it satisfies the matrix version of equation 1.1.

Definition 1.3 (Keinert [2] p.124). A function vector $\Phi : R \to C^n$ is called a multiscaling function or refinable function if it satisfies the equation

$$\phi(x) = \sum_{k=0}^{l} H_k \phi(mx - k) \qquad (1.2)$$

where $l \in N$, $H_k \in C^{n\times n}$ and $m \in R$ is the dilation factor.

Following are some of the desirable properties of a multiscaling function Φ.

Definition 1.4 (Keinert [2] p.124). The refinable function vector Φ is orthogonal if

$$< \phi(x), \phi(x-t) > = \int \phi(x)\phi(x-t)^* dt = \delta_{0t} I_n \qquad (1.3)$$

Definition 1.5 (Keinert [2] p.192). The refinable function vector Φ is symmetric if each component function φ_i, $1 \le i \le n$ is symmetric about some point a_i. That is

$\varphi_i(a_i + x) = \varphi_i(a_i - x)\ \forall x,\ 1 \leq i \leq n$
(1.4)

Definition 1.6 (Keinert [2] p.130). A function vector Φ is said to have linearly independent shifts if for all sequences of vectors $\{c_k\}$ in C^n, $\sum c_k \phi(mx-k) = 0 \Rightarrow c_k = 0 \quad \forall k$ (1.5)

Definition 1.7 (Keinert [2] p.129). The support of a function vector Φ is defined as the union of the supports of its component functions. $\text{supp}\ \phi = \cup_k \text{supp}\ \phi_k$ (1.6)

A wavelet basis of multiplicity n will be constructed from Φ which is called a Multiwavelet Basis. Theory of Multiwavelets is developed analogous to the theory of scalar wavelets and most of the results in scalar case can be directly extended to the vector case. To find a solution vector Φ for the equation 1.2, we usually switch over to the frequency domain where the above equation becomes $\hat{\phi}(\xi) = H(\xi/2)\hat{\phi}(\xi/2)$

Where $H(\xi) = \frac{1}{\sqrt{m}} \sum_{k=0}^{l} H_k e^{-ik\xi}$

which is called a Symbol Function or Mask Function. A function vector Φ generates an MRA if it is an L^2 stable, compactly supported, refinable function. Φ is a compactly supported, L^2 solution of equation 1.2 with linearly indep-endent shifts and nonzero integral only if a set of conditions called Basic Feasibility Conditions are satisfied.

Theorem 1.1 (Keinert [2] p.131). A refinable function vector $\Phi \in L^2(R)$ is compactly supported with linearly independent shifts and nonzero integral only if the following conditions are satisfied

1) H(0) has an eigenvalue 1 and all other eigenvalues are less than 1 in absolute value

2) There exists a non zero vector $y_0 \in C^n$ such

$$\sum_k y_0^* \phi(x-k) = c$$

where c is a non zero constant

3) The same vector y_0 satisfies $y_0^* H\left(\frac{2\pi t}{m}\right) = \delta_{0t} y_0^* \quad t = 0,1,\dots m-1$

4) The same vector y_0 satisfies $y_0^* \sum_k H_{pm+t} = \frac{1}{\sqrt{m}} y_0^* \quad t = 0,1,\dots m-1$

The properties of a multiscaling function is dependent on the corresponding symbol function. Thus our main focus changes to the symbol function $H(\xi)$ which is a Matrix Polynomial restricted to the unit circle in the Complex Plane. If $H(\xi)$ is the mask function then its spectral data, eigen values and generalized eigen vectors, can be found. Now, this article is organized as follows. Theory of the spectral data of matrix polynomials is discussed in section (2). We can construct a matrix polynomial from the given spectral data using the inverse representation theorem which is stated in section (3). In section (3), we have explained the necessary as well as sufficient conditions a spectral data must possess so that the corresponding matrix polynomial is the symbol function of a multiscaling function. We have explained a method to construct the symbol function H (ξ) by choosing a suitable spectral data.

2. Preliminaries: spectral data of matrix polynomials

polynomials : Let $L(\lambda) = \sum_{k=0}^{l} A_k \lambda^k \quad A_k \in C^{n\times n} \quad \lambda \in C$ (2.1) be a matrix polynomial of degree l. Then $\lambda_0 \in C$ is said to be an eigen value of $L(\lambda)$ if $\text{Det}\, L(\lambda_0) = 0$. Then there exists a non zero $x_0 \in C^n$ such that $L(\lambda_0)x_0 = 0$.

Definition 2.1. (Gohberg, Lancaster [1] p.23). The chain of vectors $x_0, x_1, \dots x_k \in C^n$ such that $x_0 \neq 0$, is a Jordan chain of length $k+1$ of $L(\lambda)$ if $\sum_{p=0}^{i} \frac{L^p(\lambda_0)}{p!} x_{i-p} = 0 \quad i = 0,1,2,\dots k$ (2.2)

Where $L^p(\lambda_0)$ is the p^{th} derivative of $L(\lambda)$ at λ_0.

A matrix polynomial is said to be regular if $\text{Det}\, L(\lambda_0) \not\equiv 0$. In this paper, we will be considering only regular matrix polynomials.

Proposition 2.1 (Gohberg, Lancaster [1] p.27). The vectors $x_0, x_1, \dots x_k \in C^n$ form a Jordan chain of the matrix polynomial $L(\lambda) = \sum_{t=0}^{l} A_t \lambda^t$ corresponding to the eigen value λ_0 iff $x_0 \neq 0$ and

$$A_0X_0 + A_1X_0J_0 + A_2X_0{J_0}^2 + \cdots + A_0X_0{J_0}^l = 0 \quad X_0 = [\,x_0\ x_1\ \dots\ x_k] \qquad (2.3)$$

Here X_0 is $n \times (k+1)$ matrix and J_0 is a Jordan block of size $(k+1)\times(k+1)$ with λ_0 on the main diagonal.

Definition 2.2 (Gohberg, Lancaster [1] p.27). A set of Jordan chains $\Phi_{j0}^{(i)}, \Phi_{j1}^{(i)}, \dots, \Phi_{j\mu_j^i-1}^{(i)} \quad j = 1,2,\dots,S_i$ of a matrix polynomial $L(\lambda) = \sum_{j=0}^{l} A_j \lambda^j$ corresponding to the eigenvalue λ_i with geometric multiplicity S_i and algebraic multiplicity α_i, is said to be a canonical set if the eigen vectors $\Phi_{j0}^{(i)}, \Phi_{j1}^{(i)}, \ldots, \Phi_{j\mu_j^i-1}^{(i)}$ are linearly independent and $\sum_{j=1}^{S_i} \mu_j^i = \alpha_i$

Consider the pair $(X_i\ J_i)$ where

$$X_i = \Big[\underbrace{\Phi_{10}^{(i)}, \ldots, \Phi_{1\mu_1^i-1}^{(i)}}_{1}, \underbrace{\Phi_{20}^{(i)}, \ldots, \Phi_{2\mu_2^i-1}^{(i)}}_{2} \ldots\ldots \underbrace{\Phi_{S_i0}^{(i)}, \ldots, \Phi_{S_i\mu_{S_i}^i-1}^{(i)}}_{S_i}\Big]$$

is a matrix of size $n \times \alpha_i$ and

$$J_i = \begin{pmatrix} J_{i1} & . & . & . & . \\ . & J_{i2} & . & . & . \\ . & . & J_{i3} & . & . \\ . & . & & \ldots & . \\ . & . & . & . & J_{iS_i} \end{pmatrix}$$

is a block diagonal matrix of size α_i . The pair of matrices (X_i, J_i) is called a Jordan apir of $L(\lambda)$ corresponding to λ_i. Now, we consider a pair of matrices $(X_F\ J_F)$ such that $X_F = (X_1\ X_2\ \ldots\ X_r)$ is a matrix of order $n \times nl$ and $J_F = \mathrm{Diag}[J_1\ J_2\ \ldots\ J_r]$ is a block diagonal matrix of order nl where each $(X_i\ , J_i)$ is a Jordan pair of $L(\lambda)$ corresponding to any finite eigenvalue λ_i. Then $(X_F\ J_F)$ is called a finite Jordan pair of $L(\lambda)$. An infinite Jordan pair $(X_\infty\ J_\infty)$ of $L(\lambda)$ is defined as the Jordan pair of $\lambda^l L(\lambda^{-1})$ corresponding to the eigen value 0.

Definition 2.3. (Gohberg, Lancaster [1] p.188). A pair of matrices

$$X = [X_1\ X_2] \text{ and } T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

where $X_1 \in C^{n\times m}, X_2 \in C^{n\times(nl-m)}$ and $T_1 \in C^{m\times m}$, $T_2 \in C^{(nl-m)\times(nl-m)}$ with $0 \le m \le nl$ is called a decomposable pair of degree l if the matrix

$$S_{l-1} = \mathrm{Col}[X_1T_1^i \quad X_2T_2^{l-1-i}]_{i=0}^{l-1} = \begin{pmatrix} X_1 & X_2{T_2}^{l-1} \\ X_1T_1 & X_2{T_2}^{l-2} \\ \ldots & \ldots \\ X_1{T_1}^{l-1} & X_2 \end{pmatrix}$$

is non singular. A pair (X, T) satisfying this property is called a decomposable pair of the regular $n \times n$ matrix polynomial $L(\lambda) = \sum_{i=0}^{l} A_i\lambda^i$ if

$$\sum_{i=0}^{l} A_iX_1{T_1}^i = 0 \quad , \quad \sum_{i=0}^{l} A_iX_2{T_2}^{l-i} = 0$$
(2.4)

Let (X_1, T_1) and (X_2, T_2) denote the finite and infinite Jordan pair of $L(\lambda)$ respectively. Then $([X_1\ X_2], T_1 \oplus T_2)$ is a decomposable pair of $L(\lambda)$ (Gohberg, Lancaster [1] p.189).

3. The Inverse Problem : We will state the inverse representation theorem of matrix polynomials and try to explore the necessary and sufficient properties a spectral data must possess so that the corresponding matrix polynomial represents a multiscaling function.

Theorem 3.1. (Gohberg,Lancaster [1] p.197). Let $(X, T) = ([X_1\ X_2], T_1 \oplus T_2)$ be a decomposable pair of degree l, and let $S_{l-2} = \mathrm{Col}[X_1T_1^i \quad X_2T_2^{l-2-i}]_{i=0}^{l-2}$

Then, for every $n \times nl$ matrix V such that the matrix $\begin{bmatrix} S_{l-2} \\ V \end{bmatrix}$ is non singular, the matrix polynomial

$$L(\lambda) = V(I - P)\big((I\oplus T_2)\lambda - (T_1\oplus I)\big)(U_0 + U_1\lambda + U_2\lambda^2 + \cdots + U_{l-1}\lambda^{l-1})$$
(3.1)

has (X, T) as its decomposable pair.

Where

$$P = (I\oplus T_2)[\mathrm{Col}(X_1T_1^i \quad X_2T_2^{l-1-i})_{i=0}^{l-1}]^{-1}\begin{pmatrix} I \\ 0 \end{pmatrix} S_{l-2}$$
(3.2)

and

$$[U_0\ U_1\ U_2 \ldots U_{l-1}] = [\mathrm{Col}(X_1T_1^i \quad X_2T_2^{l-1-i})_{i=0}^{l-1}]^{-1}$$
(3.3)

The matrix polynomial determined by $([X_1\ X_2], T_1 \oplus T_2)$ is unique up to multiplication by a non singular constant matrix Q, i.e if $L(\lambda)$ and $\tilde{L}(\lambda)$ have the same decomposable pair,

Then $\tilde{L}(\lambda) = Q \times L(\lambda)$ (3.4)

for some constant non singular matrix Q. The converse is also true, if $\tilde{L}(\lambda)$ is defined as in equation 3.4, then $L(\lambda)$ and $\tilde{L}(\lambda)$ have the same decomposable pair. Now we will state the necessary properties a Jordan pair must possess so that the corresponding matrix polynomial satisfies the Basic Feasibility Conditions (Theorem1.1).

Proposition 3.1. Let $([X_1\ X_2], T_1 \oplus T_2)$ be a given Jordan pair, then the corresponding matrix polynomial is the mask function of a compactly supported multiscaling function vector in L2 with nonzero integral and linearly independent shifts only if there exists a $n \times nl$ matrix V such that the matrix $\begin{bmatrix} S_{l-2} \\ V \end{bmatrix}$ is non singular, where $S_{l-2} = \mathrm{Col}[X_1T_1^i \quad X_2T_2^{l-2-i}]_{i=0}^{l-2}$ and

1. The matrix
$$V(I-P)\big((I\oplus T_2)-(T_1\oplus I)\big)(U_0+U_1+U_2+\cdots+U_{l-1})$$
has a simple eigen value 1 and all other eigen values are less than 1 in absolute value, where the matrices P and $[U_0\ U_1\ U_2 \dots U_{l-1}]$ are given by equations 3.2 and 3.3 respectively. Hence there exists a nonzero column vector y_0 such that
$$y_0^* V(I-P)\big((I\oplus T_2)-(T_1\oplus I)\big)(U_0+U_1+U_2+\cdots+U_{l-1})=y_0^* \qquad (3.5)$$

2. The same vector y_0 satisfies
$$y_0^* V(I-P)\big((I\oplus T_2)-(T_1\oplus I)\big)\big(U_0-U_1+U_2-\cdots+(-1)^{l-1}U_{l-1}\big)=0 \qquad (3.6)$$
In general , for $\lambda_t=e^{\frac{-2\pi i t}{m}}, t=0,1,\dots m-1$
$$y_0^* V(I-P)\big((I\oplus T_2)\lambda_t-(T_1\oplus I)\big)\big(U_0+U_1\lambda_t+U_2{\lambda_t}^2+\cdots+(\lambda_t)^{l-1}U_{l-1}\big)=\delta_{0t}y_0^* \qquad (3.7)$$

3. The vector y_0 satisfies
$$y_0^*\sum_k L_{km+t}=\frac{1}{\sqrt{m}}y_0^* \ , t=0,1,\dots m-1 \qquad (3.8)$$
where $L_k=V(I-P)\big((I\oplus T_2)U_{k-1}-(T_1\oplus I)U_k\big)$

Proof. From the inverse representation theorem 3.1, it is known that $L(\lambda)$ given by
$$L(\lambda)=V(I-P)\big((I\oplus T_2)\lambda-(T_1\oplus I)\big)(U_0+U_1\lambda+U_2\lambda^2+\cdots+U_{l-1}\lambda^{l-1})$$
has the decomposable pair $([X_1\ X_2], T_1\oplus T_2)$. From Theorem 1.1, a matrix polynomial H (ξ) is a mask function of a multiscaling function vector Φ with properties stated above only if it satisfies the Basic Feasibility Conditions.

1. H(0) must have a simple eigenvalue 1 and all other eigenvalues are less than 1 in absolute value. Consider the matrix polynomial $L(\lambda)$ corresponding to the given decomposable pair where $\lambda=e^{-i\xi}$. Now,
$L(\lambda=e^{-i\xi})|_{\xi=0}=L(1)$. Thus the matrix L(1) should have a simple eigen value 1 and all other eigen values are less than 1 in absolute value. But L(1) is given by
$$L(1)=V(I-P)\big((I\oplus T_2)-(T_1\oplus I)\big)(U_0+U_1+U_2+\cdots+U_{l-1})$$
The matrix L(1) must have a simple eigen value 1 and all other eigen values are less than 1 in absolute value. Also there must exist a non zero vector y_0 such that $y_0^*L(1)=y_0^*$
i.e.
$$y_0^*\big[V(I-P)\big((I\oplus T_2)-(T_1\oplus I)\big)(U_0+U_1+U_2+\cdots+U_{l-1})\big]=y_0^*$$

2. For the matrix polynomial H (ξ), the non zero vector y_0 satisfies
$$y_0^*H\left(\frac{2\pi t}{m}\right)=\delta_{0t}y_0^* \quad t=0,1,\dots,m-1 \qquad (3.9)$$
Take $\lambda_t=e^{\frac{-2\pi i t}{m}}, t=0,1,\dots m-1$. For the matrix polynomial $L(\lambda)$ corresponding to the given decomposable pair, equation 3.9 can be written as
$$y_0^*L(\lambda_t)=\delta_{0t}y_0^* \quad t=0,1,\dots,m-1 \qquad (3.10)$$
But
$$L(\lambda_t)=V(I-P)\big((I\oplus T_2)\lambda_t-(T_1\oplus I)\big)(U_0+U_1\lambda_t+U_2{\lambda_t}^2+\cdots+U_{l-1}{\lambda_t}^{l-1}) \qquad (3.11)$$
Combining equations 3.10 and 3.11, we get
$$y_0^*V(I-P)\big((I\oplus T_2)\lambda_t-(T_1\oplus I)\big)\big(U_0+U_1\lambda_t+U_2{\lambda_t}^2+\cdots+U_{l-1}{\lambda_t}^{l-1}\big)=\delta_{0t}y_0^*$$

4. The vector y_0 satisfies
$$y_0^*\sum_k H_{km+t}=\frac{1}{\sqrt{m}}y_0^* \ , \qquad t=0,1,\dots m-1$$
Or
$$y_0^*\sum_k L_{km+t}=\frac{1}{\sqrt{m}}y_0^* \ , \qquad t=0,1,\dots m-1$$
But
$$L(\lambda)=\sum_k L_k\,\lambda^k$$
$$=V(I-P)\big((I\oplus T_2)\lambda-(T_1\oplus I)\big)(U_0+U_1\lambda+U_2\lambda^2+\cdots+U_{l-1}\lambda^{l-1})$$
$$\Rightarrow L_k=V(I-P)\big((I\oplus T_2)U_{k-1}-(T_1\oplus I)U_k\big)$$
Thus we get
$$y_0^*\sum_k L_{km+t}=\frac{1}{\sqrt{m}}y_0^* \ , t=0,1,\dots m-1$$

Where $L_k = V(I-P)\big((I\oplus T_2)U_{k-1} - (T_1\oplus I)U_k\big)$ ■

We have to find the sufficient conditions on the spectral pair so that it represents a symbol function H (ξ) for which there exists a solution vector Φ for the refinement equation 1.5.

Definition 3.1. (Keinert [2] p.131). A matrix is said to satisfy condition E(p) if it has a p-fold non degenerate eigenvalue 1 and all other eigen values are less than 1 in absolute value

Theorem 3.2. (Keinert [2] p.220). The equation

$$\widehat{\Phi}(\xi) = H\left(\frac{\xi}{2}\right)\widehat{\Phi}\left(\frac{\xi}{2}\right)$$

corresponding to a symbol function H (ξ) has a solution vector Φ such that $\widehat{\Phi}$ is continuous at 0 with $\widehat{\Phi}(0) \neq 0$ if H (0) satisfies condition E(p).

Now we will state a sufficient condition for the existence of a solution to the multiscaling equation.

Proposition 3.2. Let $(X, T) = ([X_1\ X_2], T_1 \oplus T_2)$ be a given Jordan pair, then there exists a symbol function H (ξ) with Jordan pair (X,T) such that the corresponding multiscaling equation 1.2 has a solution vector Φ such that $\widehat{\Phi}$ is continuous at 0 with $\widehat{\Phi}(0) \neq 0$, if there exists a $n \times nl$ matrix V such that the $n \times n$ matrix

$[V(I-P)((I \oplus T_2)-(T_1\oplus I))(U_0+U_1+U_2+\ldots+U_{l-1})$

Satisfies condition E(p) and the matrix $\begin{bmatrix} S_{l-2} \\ V \end{bmatrix}$ is non singular, where

$$P = (I\oplus T_2)[\mathrm{Col}(X_1T_1^i \quad X_2T_2^{l-1-i})_{i=0}^{l-1}]^{-1}\begin{pmatrix} I \\ 0 \end{pmatrix} S_{l-2}$$

and

$$[U_0\ U_1\ U_2 \ldots U_{l-1}] = [\mathrm{Col}(X_1T_1^i \quad X_2T_2^{l-1-i})_{i=0}^{l-1}]^{-1}$$

Proof. Given that the $n \times n$ matrix

$[V(I-P)((I \oplus T_2)-(T_1\oplus I))(U_0+U_1+U_2+\ldots+U_{l-1})$

satisfies condition E(p). From equation 3.1, the matrix polynomial H (ξ) determined by $([X_1\ X_2], T_1 \oplus T_2)$ is given by

$$H(\xi) = V(I-P)\left((I\oplus T_2)e^{-i\xi} - (T_1\oplus I)\right)(U_0 + U_1e^{-i\xi} + U_2e^{-2i\xi} + \cdots + U_{l-1}\lambda^{(l-1)e^{-i\xi}})$$

where we have taken $\lambda = e^{-i\xi}$. Then

$$H(0)=[V(I-P)((I \oplus T_2) - (T_1 \oplus I))(U_0 + U_1 + U_2 + \ldots + U_{l-1})]$$

But given that the matrix

$[V(I-P)((I \oplus T_2)-(T_1\oplus I))(U_0+U_1+U_2+\ldots+U_{l-1})$

satisfies condition E(p). i.e. H(0) satisfies condition E(p). By theorem 3.2, the equation 1.2 corresponding to the symbol function H (ξ) has a solution vector Φ such that that $\widehat{\Phi}$ is continuous at 0 with $\widehat{\Phi}(0) \neq 0$■

We will construct a symbol function by selecting suitable spectral data using the Inverse representation theorem. For that, we need the following Lemma.

Lemma 3.1. (Predrag Stanimirovic, Miomir Stankovic [9]). Let A be an $m \times n$ rectangular matrix where $m > n$. Then A has a unique Moore Penrose generalized inverse $A^g = (A^T A)^{-1}A^T$ such that $A^g A = I_n$, if A is of full rank, i.e. A has rank n.

Theorem 3.3. Let $(X, T) = ([X_1\ X_2], T_1 \oplus T_2)$ be a Jordan pair such that the $nl \times nl$ matrix $(I \oplus T_2) - (T_1 \oplus I)$ is of full rank. Then there exist a symbol function H (ξ) with Jordan pair (X,T) such that the corresponding multiscaling equation 1.2 has a solution vector Φ such that $\widehat{\Phi}$ continuous at 0 with $\widehat{\Phi}(0) \neq 0$.

Proof. Given that for the Jordan pair $(X, T) = ([X_1\ X_2], T_1 \oplus T_2)$, the $nl \times nl$ matrix $(I \oplus T_2) - (T_1 \oplus I)$ is of full rank. Since $(X, T) = ([X_1 X_2], T_1 \oplus T_2)$ is a Jordan pair, the columns of $nl \times nl$ matrix

$$S_{l-1} = [\mathrm{Col}(X_1T_1^i \quad X_2T_2^{l-1-i})_{i=0}^{l-1}]$$

are independent. Then columns of the matrix ${S_{l-1}}^{-1}$ are also independent. But, we have

$$[U_0\ U_1\ U_2 \ldots U_{l-1}] = {S_{l-1}}^{-1}$$

Then the columns of

$$[U_0 + U_1 + U_2 + \cdots + U_{l-1}] = U$$

are independent. i.e. the $nl \times n$ matrix U is of full rank. Given that $(I \oplus T_2) - (T_1 \oplus I)$ is of full rank. Since the product of two full rank matrices is also of full rank, it follows that the product matrix

$$F = ((I\oplus T_2)-(T_1\oplus I))(U_0+U_1+U_2+\ldots + U_{l-1}) \tag{3.12}$$

of size $nl \times n$ is of rank n. By Lemma 3.1, there exists a unique Moore Penrose generalized inverse matrix $F^g \in C^{n\times nl}$ such that $F^g = \left(F^TF\right)^{-1}F^T$. Now, we will show that the matrix

$$\begin{pmatrix} S_{l-2} \\ F^g \end{pmatrix}$$

is non singular. For that we will prove $S_{l-2}F = 0$. Define a matrix K as

$$K = S_{l-2}\big((I \oplus T_2) - (T_1 \oplus I)\big){S_{l-1}}^{-1} \qquad (3.13)$$

i.e.

$$K = S_{l-2}\big((I \oplus T_2) - (T_1 \oplus I)\big)[U_0\ U_1\ U_2 \ldots U_{l-1}] \qquad (3.14)$$

Define the $nl \times n$ matrices $K_0\ K_1\ K_2\ \ldots\ K_{l-1}$ such that

$$K = [K_0\ K_1\ K_2\ \ldots\ K_{l-1}]$$

Then from equation 3.12 and equation 3.14, we get

$$S_{l-2}F = K_0 + K_1 + K_2 + \cdots + K_{l-1} \qquad (3.15)$$

Now

$$K = \begin{pmatrix} X_1 & X_1{T_1}^{l-2} \\ X_1T_1 & X_1{T_1}^{l-3} \\ \ldots & \ldots \\ X_1{T_1}^{l-2} & X_2 \end{pmatrix} \big((I \oplus T_2) - (T_1 \oplus I)\big) \begin{pmatrix} X_1 & X_1{T_1}^{l-1} \\ X_1T_1 & X_1{T_1}^{l-2} \\ \ldots & \ldots \\ X_1{T_1}^{l-1} & X_2 \end{pmatrix}^{-1}$$

Take $X_1{T_1}^i = A_i$ and $X_2{T_2}^i = B_i$ where $i = 0,1,2,\ldots l-1$. Then the above product matrix K will be of the from

$$K = \begin{pmatrix} A_0 - A_1 & B_{l-1} - B_{l-2} \\ A_1 - A_2 & B_{l-2} - B_{l-3} \\ \ldots & \ldots \\ A_{l-2} - A_{l-1} & B_1 - B_0 \end{pmatrix} \begin{pmatrix} A_0 & B_{l-1} \\ A_1 & B_{l-2} \\ \ldots & \ldots \\ A_{l-1} & B_0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} A_0 & B_{l-1} \\ A_1 & B_{l-2} \\ \ldots & \ldots \\ A_{l-2} & B_1 \end{pmatrix} \begin{pmatrix} A_0 & B_{l-1} \\ A_1 & B_{l-2} \\ \ldots & \ldots \\ A_{l-1} & B_0 \end{pmatrix}^{-1} - \begin{pmatrix} A_1 & B_{l-2} \\ A_2 & B_{l-3} \\ \ldots & \ldots \\ A_{l-1} & B_0 \end{pmatrix} \begin{pmatrix} A_0 & B_{l-1} \\ A_1 & B_{l-2} \\ \ldots & \ldots \\ A_{l-1} & B_0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 & . & . & 0 & -1 & 0 & 0 & 0 & . & . & . & . & . & . & 0 \\ 0 & 1 & 0 & . & . & 0 & 0 & -1 & 0 & 0 & . & . & . & . & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 & 0 & 0 & -1 & 0 & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & . & . & . & . & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & . & . & 0 & -1 \end{pmatrix}$$

$$\text{Thus } K = \begin{pmatrix} 1 & 0 & 0 & . & . & 0 & -1 & 0 & 0 & 0 & . & . & . & . & . & . & 0 \\ 0 & 1 & 0 & . & . & 0 & 0 & -1 & 0 & 0 & . & . & . & . & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 & 0 & 0 & -1 & 0 & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & . & . & . & . & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & . & . & 0 & -1 \end{pmatrix}$$

But from equation 3.15, we have

$$S_{l-2}F = K_0 + K_1 + K_2 + \cdots + K_{l-1}$$

Where

$$K_0 = \begin{pmatrix} 1 & 0 & 0 & . & . & 0 & 0 \\ 0 & 1 & 0 & . & . & 0 & 0 \\ 0 & 0 & 1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} -1 & 0 & 0 & . & . & 0 & 0 \\ 0 & -1 & 0 & . & . & 0 & 0 \\ 0 & 0 & -1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and so on. Thus, we get $K_0 + K_1 + K_2 + \ldots + K_{l-1} = 0$ i.e. $S_{l-2} F = 0$

Thus the row vectors of the matrix S_{l-2} and the column vectors of the matrix F are orthogonal, and hence are linearly independent. So we get $\begin{pmatrix} S_{l-2} \\ F^T \end{pmatrix}$ is non singular. The Moore-Penrose generalized inverse of F is given by

$$F^g = \left(F^T F\right)^{-1} F^T$$

Since $\left(F^T F\right)$ is of rank n and

$$\begin{pmatrix} S_{l-2} \\ F^T \end{pmatrix}$$

is non singular, we get

$$\begin{pmatrix} S_{l-2} \\ F^g \end{pmatrix}$$

is non singular. Define $V = E_n F^g$ where E_n is an $n \times n$ full rank diagonal matrix where 1 occurs once and other diagonal entries are less than one in absolute value. Thus E_n satisfies condition E_p for $p = 1$. Also

$$V(I - P)((I \oplus T_2) - (T_1 \oplus I))(U_0 + U_1 + U_2 + \ldots + U_{l-1}) = V(I-P)F$$

Since

$$P = (I \oplus T_2)[\mathrm{Col}(X_1 T_1^i \quad X_2 T_2^{l-1-i})_{i=0}^{l-1}]^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} S_{l-2}$$

And $S_{l-2} F = 0$, we get $PF = 0$. Thus we get

$$V(I - P)F = VF - VPF = VF$$

Then

$$VF = E_n F^g F = E_n$$

But E_n satisfies condition E_p for $p = 1$. Thus we have found V such that

$$V(I - P)((I \oplus T_2) - (T_1 \oplus I))(U_0 + U_1 + U_2 + \ldots + U_{l-1})]$$

satisfies condition E(p). By proposition 3.2, there exists a symbol function H (ξ) with Jordan pair (X,T) such that the corresponding multiscaling equation 1.2 has a solution vector Φ such that $\widehat{\Phi}$ is continuous at 0 with $\widehat{\Phi}(0) \neq 0$.■

Now we will show that for any choice of the matrix V in the above proof, the existence of a solution is not guaranteed.

Proposition 3.3. Given a Jordan pair $(X, T) = ([X_1 \quad X_2], T_1 \oplus T_2)$. Let the matrix polynomial

$$L(\lambda) = V(I - P)\left((I \oplus T_2)\lambda - (T_1 \oplus I)\right)(U_0 + U_1\lambda + U_2\lambda^2 + \cdots + U_{l-1}\lambda^{l-1})$$

having (X,T) as its Jordan pair, generate a solution Φ of equation 1.2 such that $\widehat{\Phi}$ is continuous at 0 with $\widehat{\Phi}(0) \neq 0$. Then any matrix polynomial $\tilde{L}(\lambda)$ having the same Jordan pair (X,T) may not generate such a multiscaling function.

Proof. Given that the matrix polynomial

$$L(\lambda) = V(I - P)\left((I \oplus T_2)\lambda - (T_1 \oplus I)\right)(U_0 + U_1\lambda + U_2\lambda^2 + \cdots + U_{l-1}\lambda^{l-1})$$

generates a solution Φ of equation 1.2 such that $\widehat{\Phi}$ is continuous at 0 with $\widehat{\Phi}(0) \neq 0$. Then L(1) will have an eigen value 1 [Keinert [2] Theorem(11.1) p.220]. Choose $a > 1$ such that any of the eigenvalues of L(1) not equal to $\frac{1}{a}$. Consider

$$\tilde{L}(\lambda) = \tilde{V}(I - P)\left((I \oplus T_2)\lambda - (T_1 \oplus I)\right)(U_0 + U_1\lambda + U_2\lambda^2 + \cdots + U_{l-1}\lambda^{l-1})$$

Where $\tilde{V} = aV$. Then $\tilde{L}(\lambda)$ also have the same Jordan pair (X,T). But $\tilde{L}(1)$ does not have an eigen value 1 and will have an eigen value a which is greater than 1 (By the choice of a), which violates the necessary condition that $\tilde{L}(1)$ must have an eigen value 1 to generate a multiscaling function Φ [Keinert [2] Theorem(11.1) p.220]. Thus $\tilde{L}(\lambda)$ does not generate a solution vector Φ of equation 1.2 such that $\widehat{\Phi}$ is continuous at 0 with $\widehat{\Phi}(0) \neq 0$ even though it has the same Jordan pair (X,T).■

Conclusion: We have constructed a symbol function H (ξ) by selecting a suitable Jordan pair $([X_1 \quad X_2], T_1 \oplus T_2)$ such that the corresponding multiscaling equation 1.2 has a solution vector Φ such that $\widehat{\Phi}$ is continuous at 0 with $\widehat{\Phi}(0) \neq 0$. It is worth noting that any matrix polynomial having the same Jordan pair $([X_1 \quad X_2], T_1 \oplus T_2)$ may not generate such a multiscaling function. But a suitable choice of the matrix V in the construction will guarantee a solution. The future work is to find out the additional conditions on the Jordan pair so that this solution possesses the desirable properties like symmetricity, orthogonality and compact supportedness.

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Department of Mathematics, National Institute of Technology, Calicut.