

COEFFICIENT INEQUALITY FOR A NEWLY CONSTRUCTED SUBCLASS OF CLASS OF STARLIKE ANALYTIC FUNCTIONS

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Abstract: In this paper, we will discuss a newly constructed subclass of analytic starlike functions by which we will be obtaining sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to this subclasses.

Keywords: Univalent functions, Starlike functions, Close to convex functions and bounded functions.

Introduction : Let \mathcal{A} denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1\}$. Let \mathcal{S} be the class of functions of the form (1.1), which are analytic univalent in \mathbb{E} .

In 1916, Bieber Bach ([7], [8]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$.

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and a_2^2 for the class \mathcal{S} , Fekete and Szegő[9] used Löwner’s method to prove the following well known result for the class \mathcal{S} .

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 0; \\ 1 + 2e^{-2\mu}, & \text{if } 0 \leq \mu \leq 1; \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases} \tag{1.2}$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes \mathcal{S} (See Chhichra[1], Babalola[6]).

Let us define some subclasses of \mathcal{S} .

We denote by S^* , the class of univalent starlike functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ and satisfying the condition

$$Re \left(\frac{zg(z)}{g(z)} \right) > 0, z \in \mathbb{E}. \tag{1.3}$$

We denote by \mathcal{K} , the class of univalent convex functions $h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$ and satisfying the condition

$$Re \left(\frac{zh'(z)}{h'(z)} \right) > 0, z \in \mathbb{E}. \tag{1.4}$$

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$$Re \left(\frac{zf'(z)}{g(z)} \right) > 0, z \in \mathbb{E}. \tag{1.5}$$

The class of close to convex functions is denoted by \mathcal{C} and was introduced by Kaplan [3] and it was shown by him that all close to convex functions are univalent.

$$S^*(A, B) = \left\{ f(z) \in \mathcal{A}; \frac{zf'(z)}{f(z)} < \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E} \right\} \tag{1.6}$$

$$\mathcal{K}(A, B) = \left\{ f(z) \in \mathcal{A}; \frac{(zf'(z))'}{f'(z)} < \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E} \right\} \tag{1.7}$$

It is obvious that $S^*(A, B)$ is a subclass of S^* and $\mathcal{K}(A, B)$ is a subclass of \mathcal{K} .

We introduce a new subclass as $\left\{ f(z) \in \mathcal{A}; \frac{1}{2} \left(\frac{zf'(z)}{f(z)} + \left(\frac{zf'(z)}{f(z)} \right)^{\frac{1}{\alpha}} \right) < \frac{1+z}{1-z}; z \in \mathbb{E} \right\}$ and we will

denote this class as $f(z) \in \Sigma S^*[\alpha]$. Symbol $<$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in \mathbb{E} . Then $f(z)$ is called subordinate to $F(z)$ in \mathbb{E} if there exists a function $w(z)$ analytic in \mathbb{E} satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z)); z \in \mathbb{E}$ and we write $f(z) < F(z)$.

By \mathcal{U} , we denote the class of analytic bounded functions of the form $w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1$ (1.8) It is known that $|d_1| \leq 1, |d_2| \leq 1 - |d_1|^2$ (1.9)

2. Preliminary Lemmas: For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz} \right)$ so that

$$\frac{1+w(z)}{1-w(z)} = 1 + 2c_1 z + 2(c_2 + c_1^2)z^2 + \dots \tag{2.1}$$

3. Main Results

THEOREM 3.1: Let $f(z) \in f(z) \in \Sigma S^*[\alpha]$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2\alpha}{(\alpha+1)^3} [5\alpha^2 + 10\alpha - 3 - 8\mu\alpha(\alpha+1)]; \text{ if } \mu \leq \frac{4\alpha^2 + 8\alpha - 4}{8\alpha(\alpha+1)} & (3.1) \\ \frac{2\alpha}{\alpha+1} & ; \text{ if } \frac{4\alpha^2 + 8\alpha - 4}{8\alpha(\alpha+1)} \leq \mu \leq \frac{6\alpha^2 + 12\alpha - 2}{8\alpha(\alpha+1)} & (3.2) \\ \frac{2\alpha}{(\alpha+1)^3} [8\mu\alpha(\alpha+1) - 5\alpha^2 - 10\alpha + 3] & ; \text{ if } \mu \geq \frac{6\alpha^2 + 12\alpha - 2}{8\alpha(\alpha+1)} & (3.3) \end{cases}$$

The results are sharp.

Proof: By definition of $f(z) \in f(z) \in \Sigma S^*[\alpha]$, we have

$$\frac{1}{z} \left(\frac{zf'(z)}{f(z)} + \left(\frac{zf'(z)}{f(z)} \right)^{\frac{1}{\alpha}} \right) = \frac{1+w(z)}{1-w(z)}; w(z) \in \mathcal{U}. \quad (3.4)$$

Expanding the series (3.4), we get $1 + a_2z \left(\frac{\alpha+1}{2\alpha} \right) + \frac{z^2}{2} [(2a_3 - a_2^2) \left(\frac{\alpha+1}{\alpha} \right) + \left(\frac{1-\alpha}{2\alpha^2} \right) a_2^2] + \dots = (1 + 2c_1z + 2(c_1^2 + c_2)z^2 + z^3(2c_3 + 4c_1c_2 + c_1^3) + \dots)$. (3.5)

Identifying terms in (3.5), we get

$$a_2 = \frac{4\alpha c_1}{\alpha+1} \quad (3.6)$$

$$a_3 = \left(\frac{2\alpha}{\alpha+1} \right) \left[c_1^2 + c_2 + \frac{4c_1^2}{(\alpha+1)^2} [\alpha^2 + 2\alpha - 1] \right] \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$a_3 - \mu a_2^2 = c_1^2 \left[\frac{2\alpha}{\alpha+1} + \frac{8\alpha(\alpha^2+2\alpha-1)}{(\alpha+1)^3} - \frac{16\mu\alpha^2}{(\alpha+1)^2} \right] + c_2 \left[\frac{2\alpha}{\alpha+1} \right] \quad (3.8)$$

Taking absolute value, (3.8) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \left| \frac{2\alpha}{\alpha+1} + \frac{8\alpha(\alpha^2+2\alpha-1)}{(\alpha+1)^3} - \frac{16\mu\alpha^2}{(\alpha+1)^2} \right| |c_1^2| + |c_2| \left| \frac{2\alpha}{\alpha+1} \right| \quad (3.9)$$

Using (1.9) in (3.9), we get

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{\alpha+1} + \left[\frac{2\alpha}{(\alpha+1)^3} |(5\alpha^2 + 10\alpha - 3) - 8\mu\alpha(\alpha+1)| - \frac{2\alpha}{\alpha+1} \right] |c_1|^2 \quad (3.10)$$

Case I: $\mu \geq \frac{5\alpha^2+10\alpha-3}{8\alpha(\alpha+1)}$. (3.10) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{\alpha+1} + \frac{2\alpha}{(\alpha+1)^3} [8\mu\alpha(\alpha+1) - (6\alpha^2 + 12\alpha - 2)] |c_1|^2 \quad (3.11) \quad (3.11)$$

Subcase I (a): $\mu \geq \frac{6\alpha^2+12\alpha-2}{8\alpha(\alpha+1)}$. Using (1.9), (3.11) becomes

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{(\alpha+1)^3} [8\mu\alpha(\alpha+1) - 5\alpha^2 - 10\alpha + 3]. \quad (3.12)$$

Subcase I (b): $\mu \leq \frac{6\alpha^2+12\alpha-2}{8\alpha(\alpha+1)}$. We obtain from (3.11)

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{\alpha+1} \quad (3.13)$$

Case II: $\mu \leq \frac{5\alpha^2+10\alpha-3}{8\alpha(\alpha+1)}$

Proceeding as in case I, we get

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{\alpha+1} + \frac{2\alpha}{(\alpha+1)^3} [4\alpha^2 + 8\alpha - 4 - 8\mu\alpha(\alpha+1)] |c_1|^2. \quad (3.14)$$

Subcase II (a): $\mu \leq \frac{4\alpha^2+8\alpha-4}{8\alpha(\alpha+1)}$ Using (1.9),

(3.14) takes the form

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{(\alpha+1)^3} [5\alpha^2 + 10\alpha - 3 - 8\mu\alpha(\alpha+1)] \quad (3.15)$$

Subcase II (b): $\mu \geq \frac{4\alpha^2+8\alpha-4}{8\alpha(\alpha+1)}$

Proceeding as in subcase I (a), we get

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{\alpha+1} \quad (3.16)$$

Combining (3.12), (3.13), (3.15) and (3.16), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by

$$f_1(z) = z \left(1 + \frac{59\alpha^3 - 15\alpha^2 - 7\alpha + 3}{4\alpha(\alpha + 1)^2} z \right)^{\frac{64\alpha^3}{59\alpha^3 - 15\alpha^2 - 7\alpha + 3}}$$

Extremal function for (3.2) is defined by $f_2(z) = z(1 + z^2)^{\frac{2\alpha}{\alpha+1}}$.

Corollary 3.2: Putting $\alpha = 1$, in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq \frac{1}{2}; \\ 1 & \text{if } \frac{1}{2} \leq \mu \leq 1; \\ 4\mu - 3, & \text{if } \mu \geq 1 \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent convex functions.

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