FEKETE-SZEGÖ INEQUALITY FOR A NEW SUBCLASS

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Abstract: We will describe a subclass of p-valent analytic functions in this paper and will obtain sharp upper bounds of the functional $|a_{p+2} - \mu a_{p+1}^2|$ for the analytic function $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, |z| < 1$ belonging to this subclass.

Keywords: Univalent functions, Starlike functions, Close to convex functions and bounded functions.

Introduction : Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([7], [8]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$, Fekete and Szegö[9] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if\ \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \leq \mu \leq 1; \\ 4\mu - 3, if \mu \geq 1. \end{cases}$$

(1.2)

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (See Chhichra[1], Babalola[6]).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

$\in \mathcal{A}$ and satisfying the condition

$$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \qquad (1.3)$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z$$

$\in \mathcal{A}$ and satisfying the condition

$$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \qquad (1.4)$$

p-Valent Function:

Multivalent functions and in particular p-valent functions, are a generalization of univalent functions. In the study of univalent functions, one of the fundamental problems is whether there exists a univalent mapping from a given domain E onto a given domain D. A necessary condition for the existence of such a mapping is that E and D have equal degrees of connectivity. If E and D are simply-connected domains whose boundaries contain more than one point, then this condition is also sufficient and the problem reduces to mapping a given domain onto a disc. In this connection, a special role is played in the theory of univalent functions on simply-connected domains by the S, class of functions f that are regular and univalent on the unit disc $E = \{z: |z| < 1\}$, normalized by the conditions $f(0) = 0, f'(0) = 1$, and having the expansion

$$f(z) = z + a_2 z^2 + a_3 z^3 + - - -, z \in E$$

In the case of multiply-connected domains, mappings of a given multiply-connected domain onto so-called canonical domains are studied. In particular, p-valent functions can be defined as follow:

Let $\mathcal{A}_p$ (p is a positive integer) denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$$

which are analytic in the unit disc E. Clearly, $\mathcal{A}_1 = \mathcal{A}$. A function $f(z) \in \mathcal{A}_p$ is said to be p-valent in E if it assumes no value more than p times in E.

p-Valent Starlike Function:

A function $f(z) \in \mathcal{A}_p$ is said to be a p-valent starlike function in E if there exists a positive real number ρ such that

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0$$

and

$$\int_0^{\pi}\left[Re\left(\frac{zf'(z)}{f(z)}\right)\right]d\theta = 2p\pi, z = re^{i\theta} for$$

$$\rho < |z| < 1.$$

We denote the class of p-valent starlike functions by S_p^*. By $S_p^*(\beta)$, we denote the class of functions $f(z) \in \boldsymbol{\mathcal{A}_p}$ satisfying the condition

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \beta; 0 \le \beta < p, z \in E$$

Note: p-valent starlike functions are also called p-valently starlike functions.

$f(z) \in S_p^*(\beta)$ is called p-valently starlike function of order β.

We introduce a new subclass as $\left\{\mathbf{f(z)} \in \boldsymbol{\mathcal{A}}_\mathbf{p}; \frac{\left[\mathbf{z\{zf'(z)\}'}\right]'}{\mathbf{p\{zf'(z)\}'}} \prec \frac{\mathbf{1+z}}{\mathbf{1-z}}; \mathbf{z} \in \mathbb{E}\right\}$ and we will denote this class as $f(z) \in \mathcal{H}_p^*$.

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form $w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1.$ (1.5)

It is known that $|d_1| \le 1, |d_2| \le 1 - |d_1|^2$. (1.6)

2. Preliminary Lemmas: For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz}\right)$ so that

$$\frac{1+w(z)}{1-w(z)} = 1 + 2c_1 z + 2(c_2 + c_1^2)z^2 + --- \quad (2.1)$$

Here $|c_1| \le 1, |c_2| \le 1 - |c_1|^2$ (2.2)

3. MAIN RESULTS

THEOREM 3.1: Let $f(z) \in \mathcal{H}_p^*$, then

$$|a_{p+2} - \mu a_{p+1}^2| \le \begin{cases} \frac{p^3(2p+1)}{(p+2)^2} - \frac{4up^6}{(p+1)^4} \ if \mu \le \frac{(p+1)^4}{2p^2(p+2)^2} & (3.1) \\ \frac{p^3}{(p+2)^2} \ if \ \frac{(p+1)^4}{2p^2(p+2)^2} \le \mu \le \frac{(p+1)^5}{2p^3(p+2)^2} & (3.2) \\ \frac{4up^6}{(p+1)^4} - \frac{p^3(2p+1)}{(p+2)^2} \ if \ \mu \ge \frac{(p+1)^5}{2p^3(p+2)^2} & (3.3) \end{cases}$$

The results are sharp.

Proof: By definition of $f(z) \in \mathcal{H}_p^*$, we have

$$\frac{\left[\mathbf{z\{zf'(z)\}'}\right]'}{\mathbf{p\{zf'(z)\}'}} = \frac{1+w(z)}{1-w(z)}; w(z) \in \mathcal{U}. \quad (3.4)$$

Expanding the series (3.4), we get

$$p^3z^{p-1} + a_{p+1}(p+1)^3z^p + a_{p+2}(p+2)^3z^{p+1} + --- = (1 + 2c_1z + (2c_2 + 2c_1^2)z^2 + ---)(p^3z^{p-1} + a_{p+1}(p+1)^2p z^p + a_{p+2}(p+2)^2 p z^{p+1} + ---) \quad (3.5)$$

Identifying terms in (3.5), we get

$$a_{p+1} = \frac{2c_1p^3}{(p+1)^2} \quad (3.6)$$

$$a_{p+2} = \frac{2{c_1}^2p^4 + (c_2 + c_1^2)p^3}{(p+2)^2} \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$a_{p+2} - \mu a_{p+1}^2 = \frac{2{c_1}^2p^4 + (c_2 + c_1^2)p^3}{(p+2)^2} - \mu\frac{4{c_1}^2p^6}{(p+1)^4} \quad (3.8)$$

Taking absolute value, (3.8) can be rewritten as

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{|c_2|p^3}{(p+2)^2} + \left|\frac{2p^4+p^3}{(p+2)^2} - \frac{4\mu p^6}{(p+1)^4}\right||c_1|^2 \quad (3.9)$$

Using (1.6) in (3.9), we get

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{p^3(1-|c_1|^2)}{(p+2)^2} + \left|\frac{2p^4+p^3}{(p+2)^2} - \frac{4\mu p^6}{(p+1)^4}\right||c_1|^2 \quad (3.10)$$

Case I: $\mu \le \frac{(2p+1)(p+1)^4}{4p^3(p+2)^2}$

(3.10) can be rewritten as

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{p^3(1-|c_1|^2)}{(p+2)^2} + \left(\frac{2p^4+p^3}{(p+2)^2} - \frac{4\mu p^6}{(p+1)^4}\right)|c_1|^2$$

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{p^3}{(p+2)^2} + \left(\frac{2p^4}{(p+2)^2} - \frac{4\mu p^6}{(p+1)^4}\right)|c_1|^2 \quad (3.11)$$

Subcase I (a): $\mu \le \frac{(p+1)^4}{(p+2)^2 2p^2}$

Using (2.2), (3.11) becomes

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{p^3(2p+1)}{(p+2)^2} - \frac{4\mu p^6}{(p+1)^4} \quad (3.12)$$

Subcase I (b): $\mu \ge \frac{(p+1)^4}{(p+2)^2 2p^2}$. We obtain from (3.11)

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{p^3}{(p+2)^2} \quad (3.13)$$

Case II: : $\mu \ge \frac{(2p+1)(p+1)^4}{4p^3(p+2)^2}$

Preceding as in case I, we get

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{p^3}{(p+2)^2} - \left(\frac{p^3(2p+2)}{(p+2)^2} - \frac{4\mu p^6}{(p+1)^4}\right)|c_1|^2 \quad (3.14)$$

Subcase II (a): $\mu \ge \frac{(p+1)^5}{2(p+2)^2p^3}$

(3.14) takes the form $|a_{p+2} - \mu a_{p+1}^2| \leq \frac{p^3}{(p+2)^2} - \left(\frac{p^3(2p+2)}{(p+2)^2} - \frac{4\mu p^6}{(p+1)^4}\right)$

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{4\mu p^6}{(p+1)^4} - \frac{p^3(2p+1)}{(p+2)^2} \quad (3.15)$$

Subcase II (b): $\mu \leq \frac{(p+1)^5}{2(p+2)^2 p^3}$

Preceding as in subcase I (b), we get

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{p^3}{(p+2)^2} \quad (3.16)$$

Combining (3.12),(3.13), (3.15) and (3.16), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by

$$f_1(z) = \left(1 - \frac{5p^2+4p+1}{(p+2)^2}\right)^{-\frac{2p^3(p+2)^2}{(p+1)^2(5p^2+4p+1)}}$$

Extremal function for (3.2) is defined by

$$f_2(z) = (1+z^2)^{\frac{p^3}{(p+2)^2}}$$

Corollary 3.2: Putting $p = 1$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{3} - \frac{\mu}{4} \ if \ \mu \leq \frac{8}{9} \\ \frac{1}{9} \ if \ \frac{8}{9} \leq \mu \leq \frac{16}{9} \\ -\frac{1}{3} + \frac{\mu}{4} \ if \ \mu \geq \frac{16}{9} \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent convex functions.

References:

1. *Ashutosh S. ingole,* An Advance Algorithm for Task Management on Activity; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 2 Issue 2 (2013), Pg 623-629
2. Chichra, P. N., “New subclasses of the class of close-to-convex functions”, Procedure of American Mathematical Society, 62, (1977), 37-43.
3. Goel, R. M. and Mehrok, B. S., “A subclass of univalent functions”, Houston Journal of Mathematics, 8, (1982), 343-357.
4. Kaplan, W., “Close-to-convex schlicht functions”, Michigan Mathematical Journal, 1, (1952), 169-185.
5. *Reny George,* Some Fixed Point Results of Presic Type in B-Metric Spaces; Mathematical Sciences International Research Journal ISSN 2278 – 8697 Vol 2 Issue 2 (2013), Pg 208-213
6. Keogh, S. R. and Merkes, E. R., “A Coefficient inequality for certain classes of analytic functions”, Procedure of American Mathematical Society, 20, (1989), 8-12.
7. K. Löwner, “Uber monotone Matrixfunktionen”, Math. Z., 38, (1934), 177-216.
8. Kunle Oladeji Babalola, “The fifth and sixth coefficients of α-close-to-convex functions”, Kragujevac J. Math., 32, (2009), 5-12.
9. L. Bieberbach, “Uber einige Extremal probleme im Gebiete der konformen Abbildung”, Math. Ann., 77(1916), 153-172.
10. K.K.Suresh, S.Umamaheswari, Optimal Design of Bayesian Skip Lot Sampling Plan 2; Mathematical Sciences international Research Journal ISSN 2278 – 8697 Vol 3 Issue 2 (2014), Pg 587-591
11. L. Bieberbach, “Uber die Koeffizientem derjenigen Potenzreihen, welche eine schlithe Abbildung des Einheitskreises vermitteln”, Preuss. Akad. Wiss. Sitzungsb., (1916), 940-955.
12. M. Fekete and G. Szegö, “Eine Bemerkung über ungerade schlichte Funktionen”, J. London Math. Soc. 8 (1933), 85-89.
13. R. M. Goel and B. S.Mehrok, “A coefficient inequality for a subclass of closeto-convex functions”, Serdica Bul. Math. Pubs., 15, (1989), 327-335.
14. Z. Nehari, “Conformal Mapping”, McGraw-Hill, Comp., Inc., New York, (1952).

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